Mathematics 41C - 43C

Mathematics Department
Phillips Exeter Academy
Exeter, NH
August 2024

41C

42C

43C

Final Projects

Appendix

Chemistry Laboratories

To the Student, from Fellow Students

Introduction: Approximately 320 new Exeter students annually enroll in a math course... with only a small, savvy, and select few that enroll in *Calculus - A Lab Approach*, or what we just call "Lab Calculus." As a new student in Lab Calculus, you will quickly realize the distinct methods of a different kind of curriculum. Some students are thrown off by the mathematics program at Exeter when they initially arrive and the same may occur when taking Lab Calculus. It is true; the class is different from the more trodden pathways of Exeter math. But we encourage all new students to come to the Lab Calculus table with a clear mind. You may not grasp, understand, or even like Lab Calculus at first, but there is no doubt that you will complete Lab Calculus thinking it was the best course selection you have ever made at Exeter. This brief guide you are reading was written by Lab Calculus alums and updated by instructors, with the intent of briefly preparing you for the task you have embarked upon. It includes tips for survival, testimonials of how we felt when entering the classroom, and aspects of math that we would have liked to have known before we felt overwhelmed. Hopefully, this guide will ease your transition at Exeter. We wish you the best of luck. *Vive Lab Calculus!* *–Charlie Holtz '23*

Contents: Members of the PEA Mathematics Department have written the material in this book. As you work through it, you will be exposed to, learn, and apply many of the foundational ideas of calculus. There is no Chapter 5 or a set section on integrating or deriving. While the Lab Calculus curriculum is problem-based, like much of Exeter mathematics, the main difference lies in how knowledge, gleaned through assigned problems, is applied and assessed. Lab Calculus is more mini-topic-centered, using frequent written reports to solidify and practice the curriculum. Calculus techniques, strategies, and theorems are introduced as you work through problem sets and are clarified with the completion of lab reports. You must keep appropriate notes for your records—there are no boxes containing essential theorems.

Problem solving: In order to succeed in this course, your ability to comprehend, apply, and solve the assigned homework problems will be crucial. The Lab Calculus learning approach emphasizes an efficient understanding of mathematical concepts, which should then be synthesized in your lab reports. Regularly tackling homework problems will enhance your problem-solving abilities while preparing you for lab assignments. Maintaining a digital or physical notebook to record answers, solutions, and notes on key ideas within each subject, is highly recommended. For instance, jotting down derivative or antiderivative rules will aid in memorizing them until they become second nature. Approach each problem with enthusiasm and curiosity. If you make mistakes or do not know the answer, persevere and try again. It is important to note that many problems require a return to fundamental concepts, so ensure you grasp the general principles of each subject. Do not hesitate to seek assistance from your teacher or classmates. Cooperative learning is encouraged and you will find that others often have the same questions as you. Remember, the techniques employed in problem-solving, the adjustments in approach, the methods used to validate solutions, and effective communication skills, are all equally significant in reaching the correct answers.

Technology: Many problems in this book require using technology (graphing calculators, computer software, or tablet applications) to solve specific problems and labs. You are encouraged to use technology to explore and formulate, and test conjectures. Keep the following guidelines in mind: write before you calculate so that you will have a clear record of what you have done; be wary of rounding mid-calculation; pay attention to the degree of accuracy requested; try to sketch all graphs before using a tool; and be prepared to explain your method to your classmates. Many resources are available online if you need to learn how to perform a needed action, but beware of substituting online research for learning through self-discovery.

Finally, unlike other math classes at Exeter, Lab Calculus utilizes technology to a higher degree. Geogebra files, spreadsheet models, projectile motion simulators, and Desmos are integral to understanding the many labs you will complete throughout the term.

Standardized testing: Standardized tests like the SAT, ACT, and Advanced Placement tests require calculators for specific problems. Although Lab Calculus uses a variety of tools, it is still useful for students to know how to use a hand-held graphing calculator to perform specific tasks. Among others, these tasks include: graphing, finding minima and maxima, creating scatter plots, regression analysis, and general numerical calculations. Lab calculus, usually taken in the upper or senior year, provides valuable practice before the college entrance exams.

Homework: Putting your best effort into the homework problems each night is critical, although you may need help to complete them all thoroughly. When attempting the homework, remember to make connections by asking yourself, "What parts of this problem are similar to problems we have done in the past? What parts are new?" If you feel stuck, start by identifying what the question asks you to do, even if you are unsure how to do it. Is the question asking you to explain something, or will your answer be a value found through mathematical calculation? Then look to see if there is information you could use to create a diagram or graph to help you better understand the problem. If you work through the problem to the best of your ability but are still confused, that is ok! It is better to get something in your notes than nothing. Write yourself a note about where in the problem your confusion lies and then be sure to ask questions when the issue is presented in class. This process keeps the class momentum going and is critical to a collaborative learning environment.

Organization is also essential when completing your homework! We suggest you keep all of your work for this class in one place (ex: a notebook, binder, tablet, etc.). Write the date at the beginning of each night of homework, and be sure to write the number of every problem you do before you begin it so you can go back and reference your work later in the term or year. Leave ample space for in-class corrections and connections to your work, and make these additions in a separate color from your homework so you can distinguish what you did for homework and what you did in class. Determining a system early on that works for you in lab calculus will ensure you spend more of your time in class actually learning than wasting time with organizational issues!

Going to the board: It is imperative to go to the board to put up homework problems. Homework problems are put up on the board at the beginning of class, and then they are discussed in class. If you regularly put problems up on the board, your mathematical fluency and confidence in upcoming lab reports will grow. You are encouraged to collaborate to put up challenging problems.

Student Testimonials

After a year of struggling with Exeter math, I felt as though I only understood concepts about 50% and did poorly on tests. As a result, I found Lab Calculus, which has been an incredibly positive experience. The pace of the class and the order of the material covered varies from the other calculus clases at Exeter in a way that makes a lot of sense for learning based on understanding and building upon former concepts. Beyond that, labs force you to understand the material by requiring you to explain it in words. This method of demonstrating understanding is, in my experience, more directly correlated with true understanding than a traditional test that you could do poorly on because of stress, working at a different pace than your classmates, or any other factors unrelated to how much you actually understand the material. The class is set up so that rather than feel that each problem or problem set is an isolated concept unrelated to any learning we have done previously, everything is connected. I could genuinely explain to you how the learning of the whole year, beginning with the rate of change graphs, built on itself to create our understanding of far more complex spring terms concepts such as integration as accumulation or geometric probabilities. The intense focus on connected learning and deep understanding is the only reason I understand calculus, and I am so grateful for this course! *–Agatha Prairie '23*

I loved every part of Lab Calculus. Being a person who was not necessarily the best at Exeter math and really struggled with testing and testing anxiety, this course helped me understand that I could enjoy math and problem-solving through synthesizing and being patient with myself and my learning. I really enjoyed specifically doing the labs each week because I felt like every week was a new challenge and a new concept that I needed to understand. Writing it down and summarizing it in a lab was the best way to explain my learning. That way, I ensured that I understood the material properly. The course and the curriculum are amazing. I really enjoyed the class environment and the cooperative teaching style required of students. It was the best math class I have ever taken at Exeter and my entire life. And I would 100% recommend peopletake it if they are not super confident in their math skills or if they feel like they might need a little space to make math learning their own. *–Alexa Murat '23*

Lab Calculus is a different kind of calculus. A better kind of calculus. I have loved every moment of Lab Calculus, not to say it is easy. The class is designed to be highly collaborative - lab reports and class participation are team endeavors. Both of those aspects are what make Lab Calculus so impactful and challenging. It can be hard to divvy up parts of a lab to different team members, and on some days, class participation can be lacking. However, on the flip side, through collaborative creation of lab reports - through deep consideration of the problems, through asking clarifying questions of your peers, through explaining concepts to a friend not yet at full understanding, through presentations of labs - I have received the most complete mathematical understanding in a math class, ever. In that way, Lab Calculus is about how much work you individually put into each problem every night, each question you ask in class, each contribution you make to labs, and everyone else's participation. Lab Calculus is a beautiful Exeter experiment that went wonderfully right. I am excited you are taking this class and wish you the best of luck. Please enjoy a few Lab Calculus-specific bullet points of wisdom.

- Do not wait till the last minute to start a lab.
- Read the entire lab before starting.
- Do not get mad at Desmos/Geogebra; they are just pieces of software.
- Save Desmos files.
- Make sure all lab partners understand the lab.
- Keep good notes; early concepts come back.
- You are fairly compensated for your consistent effort in Lab Calculus.
- Remember your derivative rules.
- Be nice to your teacher; they are very helpful.
- Come to class prepared with energy, questions, and a genuine desire to learn.
- Calculate that (lab) Calculus! *–Charlie Holtz '23*

Derivative Rule	**Function**	**Derivative**
Power Rule	$f(x) = x^n$	
Constant Multiple Rule	$f(x) = c * g(x)$	
Sum/Difference Rule	$f(x) = g(x) + h(x)$	
Product Rule	$f(x) = g(x) * h(x)$	
Quotient Rule	$f(x) = \frac{g(x)}{h(x)}$	
Chain Rule	$f(x) = g(h(x))$	

Function	**Derivative**
$a(x) = c$	
$b(x) = mx + b$	
$c(x) = sin(x)$	
$d(x) = cos(x)$	
$f(x) = tan(x)$	
$g(x) = ln(x)$	
$h(x) = e^x$	
$j(x) = b^x, b > 0$	
$k(x) = log_b(x), b > 0$	

Mathematics 41C

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P1.1 Consider a spreading zombie apocalypse. What information would you need to know to be able to build a model that predicted the spread of zombies through the world? Work as a class to come up with a list of information that you require.

P1.2 Sort your list from the previous question by which group it references. For example, you may have groups such as Healthy, Zombie, Precursor Zombie, etc.

P1.3 For each of the groups in your zombie apocalypse, sketch a plausible graph of that population versus time, assuming that the apocalypse begins with a single zombie in a population of 10,000 people. What would be the same amongst all of the graphs in your class? What might be different? Why?

P1.4 A population could be modeled by $P(t) = 1.1^t$, where t is measured in days, and $P(t)$ gives the population (in thousands) on that day. Writing $P(2)$ gives the population on the second day, and $P(2) = 1.21$ means that the population on the second day is 1,210. Find values for the following and explain what they mean in the context of this model:

(a) $P(0)$

(b) $P(10)$

(c) $P(t) = 575$

P1.5 Another way to think about a population function is to use a table. Use the table below to find the indicated values, and explain their relevance in the context of a population:

Year	50	100	150	200	250	300
Population	137	162	187	212	236	259

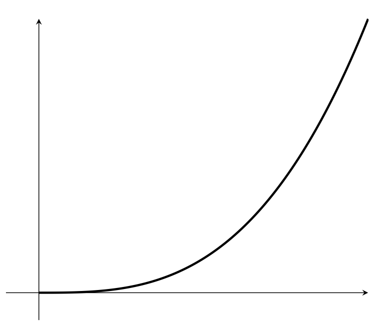
(a) $P(50)$

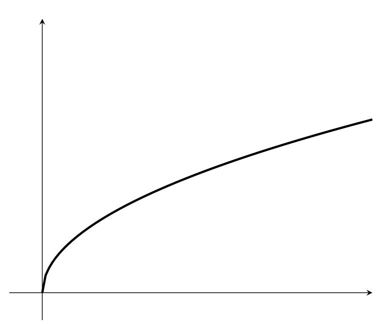
(b) $P(y) = 236$

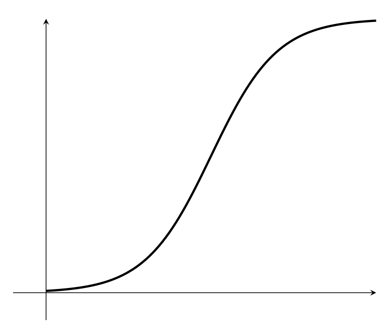
(c) $P(100) - P(50)$

(d) $\dfrac{P(100) - P(50)}{P(50)}$

P1.6 Below are three population versus time graphs. Write stories that match the graphs.







P1.7 After conversation as a class, use a spreadsheet program to generate data that matches one of the situations from the previous problem.

P1.8 Study the spreadsheet work shown below and explain what is happening in each column.

	A	B
1	Time	Population
2	0	30
3	1	43
4	2	57.3
5	3	73.03
6	4	90.333
7	5	109.3663
8	6	130.30293
9	7	153.33322
10	8	178.66655
11	9	206.5332
12	10	237.18652
13	11	270.90517
14	12	307.99569
15	13	348.79526
16	14	393.67478
17	15	443.04226
18	16	497.34649
19	17	557.08114
20	18	622.78925
21	19	695.06818
22	20	774.57499

	A	B
1	Time	Population
2	0	30
3	1	=1.1*B2+10
4	2	=1.1*B3+10
5	3	=1.1*B4+10
6	4	=1.1*B5+10
7	5	=1.1*B6+10
8	6	=1.1*B7+10
9	7	=1.1*B8+10
10	8	=1.1*B9+10
11	9	=1.1*B10+10
12	10	=1.1*B11+10
13	11	=1.1*B12+10
14	12	=1.1*B13+10
15	13	=1.1*B14+10
16	14	=1.1*B15+10
17	15	=1.1*B16+10
18	16	=1.1*B17+10
19	17	=1.1*B18+10
20	18	=1.1*B19+10
21	19	=1.1*B20+10
22	20	=1.1*B21+10

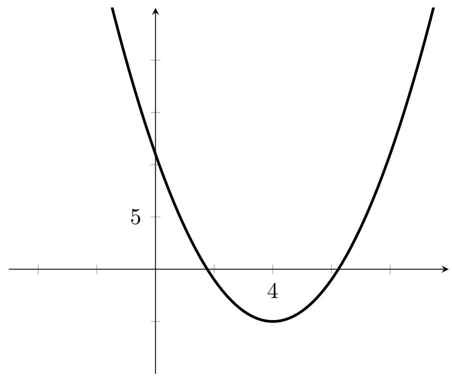
P1.9 Big Red Taxi charges a fixed amount of \$2.40 plus \$3.10 per mile. Big Blue Taxi only charges for mileage at a rate of \$3.50 per mile. Write equations expressing the amount of the fare as a function of the distance traveled for each of the cabs, and sketch graphs of the functions on the same set of axes. What is common to these two functions? What is different? What can you say about the rate of change of each fare function?

P1.10 A college savings account grows at an annual rate of 4.5%. Write an equation expressing the amount in the account t years after an initial deposit of \$5000.00, and obtain a graph of the function. Compare this function with the functions in #P1.9. How are they similar? How are they different? [Hint: consider what is constant in each function.]

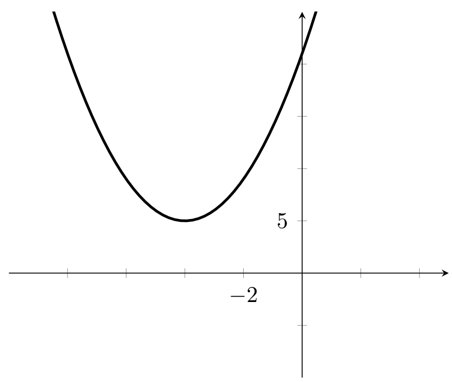
P1.11 A function f is called *even* if it has the property that $f(-x) = f(x)$ for all x-values in the domain. What does this property tell us about the appearance of the graph of $y = f(x)$? Show that $C(x) = \dfrac{2^x + 2^{-x}}{2}$ is an even function. Give other examples of even functions.

P1.12 A function f is called *odd* if it has the property that $f(-x) = -f(x)$ for all x-values in the domain. What does this property tell us about the appearance of the graph $y = f(x)$? Show that $C(x) = \dfrac{2^x - 2^{-x}}{2}$ is an odd function. Give other examples of odd functions.

P1.13 The graph of $y = g(x)$ is shown below. Knowing that $g(x)$ is a translation of $f(x) = x^2$, write an equation for $y = g(x)$ in terms of $f(x)$.



P1.14 (Continuation) Write an equation for the graph shown below in terms of $g(x)$.



P1.15 After being dropped from the top of a tall building, the height of an object is described by $h(t) = 400 - 16t^2$, where h is in feet and t is the time in seconds.

(a) Sketch a graph that shows the height on the vertical axis and time on the horizontal axis. Paying attention to the scale, label the h-intercept with an A and the t-intercept with a B.

(b) Draw line segment AB and find its slope. What does the slope of AB tell you about the falling object?

(c) Let C be the point when $t = 2$ and D be the point when $t = 2.1$, and draw the line segment CD. What does the slope of CD tell you about the falling object?

(d) If the height of the object dropped from the tall building were given by $H(t) = 450 - 16t^2$ instead of $h(t) = 400 - 16t^2$ how would your answers to (b) and (c) change, if at all?

P1.16 Given $f(x) = 3x + 4x^2$.

(a) Write expressions for $f(x-1)$ and $f(2x)$. Simplify if you think it is informative, and sketch all three curves.

(b) Find x if $f(x) = 7$ and compare your approach with your classmates.

(c) Find x if $f(x-1) = 7$ and $f(2x) = 7$, and again compare approaches.

P1.17 Given $g(x) = x + \dfrac{3}{2-x}$

(a) Solve the equation $g(x) = 0$ for x.

(b) Use your result from $g(x) = 0$ to solve for $g(x+1) = 0$.

P1.18 How many solutions are there to the equation $x^2 = 2^x$? Find all of them.

P1.19 (Continuation) From the solutions, which x-value is largest? Which is largest for $x^2 = 1.5^x$? What about $x^2 = 1.1^x$?

P1.20 Suppose we compare the graphs of $y = x^2$ and $y = 1.01^x$. The first is a parabola, and the second is an exponential function with a growth rate of 1%. Which function is greater for $x = 100$? $x = 1000$? $x = 10,000$? Based on these observations, which function grows faster in the long run?

P1.21 (Continuation) Make a conjecture about which is the faster growing function: a power function $y = x^n$, where n is a positive integer, or $y = b^x$ where $b > 1$.

P1.22 Which is best, to have money in a bank that pays 9 percent annual interest, one that pays 9/12 percent monthly interest, or one that pays 9/365 percent daily interest? A bank is said to *compound* its annual interest when it applies a fraction of its annual interest to a fraction of a year.

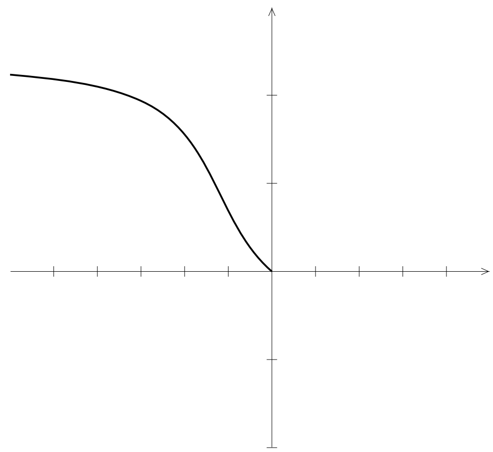
P1.23 (Continuation) Inflation in the country of Syldavia has reached alarming levels. Many banks are paying 100 percent annual interest, some banks are paying 100/12 percent monthly interest, a few are paying 100/365 percent daily interest, and so forth. Trying to make sense of all of these promotions, Milou decides to graph the function E, given by $E(x) = \left(1 + \dfrac{1}{x}\right)^x$. What does this graph reveal about the sequence $v_n = \left(1 + \dfrac{1}{n}\right)^n$, where n is a positive integer? Calculate the specific values: v_1, v_{12}, v_{365} and $v_{31536000}$.

P1.24 (Continuation) Numerical and graphical evidence from #P1.23 suggests that the expression $\left(1 + \dfrac{1}{n}\right)^n$ approaches a particular number as n gets larger. In fact, it does, and this number is so important that a special letter is reserved for it (as is done for π). We define e to be the number the expression $\left(1 + \dfrac{1}{n}\right)^n$ approaches. From the previous problem, write a good approximation for e to five decimal places. Use a calculator determine what value the expression $\left(1 + \dfrac{0.09}{n}\right)^n$ approaches as n gets large. What does this value mean in the context of #P1.22?

P1.25 What single word describes a function f that has the property $f(x) = f(x + 60)$ for all values of x?

P1.26 Part of the graph of $y = f(x)$ is shown below. Draw the rest of the graph given

(a) f is an even function, **(b)** f is an odd function.



P1.27 Without a calculator and on the same set of axes, graph both $f(x) = x^3$ and $y = f(x - 1) + 2$.

P1.28 Find a function for which $f(x + a) = f(x)f(a)$ for all numbers x and a.

P1.29 Is the graph of $k(x) = x - x^2$ an even function, an odd function, or neither? How do you know?

P1.30 Is the graph of $t(x) = x^3 + x$ an even function, an odd function, or neither? How do you know?

In the first problem set, you have spent time exploring what happens as zombies spread through a population. The following laboratory will help you to delve deeper into this idea, and to practice what it means to capture your understanding in writing.

Prelaboratory 1: Modeling Changing Populations

Using Excel (or Google sheets or some other spreadsheet software), create a data table that models what happens to the caterpillar and butterfly population (on a small island) if:

- It starts with 2 caterpillars and 0 butterflies on Day 0
- Each day, 2 caterpillars are born (i.e., there are 4 caterpillars on Day 1)
- After being alive for 6 days, each caterpillar forms a cocoon (i.e., on Day 6, the first two caterpillars from Day 0 become cocooned)
- After being in a cocoon for 4 days, it becomes a butterfly (i.e., on Day 10, the first two caterpillars are now butterflies)

Create separate columns for the number of caterpillars, the number of cocoons, and the number of butterflies on the island each day. Set up formulas within your spreadsheet to call on cells from different columns. For example, on Day 6, where are the two cocooned caterpillars coming from? Use your spreadsheet to determine the population of caterpillars and butterflies on Day 20.

Lab 1: Modeling the Zombie Apocalypse

L1.1 Consider a population of 1,000 people where just one person is a zombie. Sketch graphs of the various groups over time based on the previous work you have done in this course.

L1.2 If the number of zombies starts with ten instead of one, how does each of your graphs change?

L1.3 There are many aspects of these models that are overly simplistic. Pick one aspect and adjust your model accordingly. Explain any assumptions you make along the way. Based on your changes, what do your graphs look like?

L1.4 Refine your graphs by creating a spreadsheet that calculates the changes in population based on your particular model. If the resulting data differs from your projections, explain why.

L1.5 Write a paragraph summarizing what you have discovered in this laboratory. Be thoughtful about your integration of prose, spreadsheet work, and graphs.

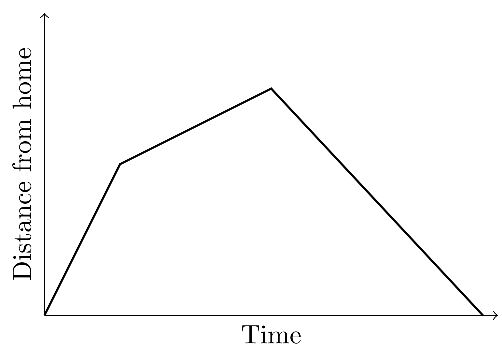
P2.1 The point $(4, 16)$ is on the graph of $f(x) = x^2$. Treating (x, x^2) as an arbitrary point on the graph of $f(x) = x^2$, the fraction $\dfrac{x^2 - 16}{x - 4}$ represents the slope between two points on the graph. Find the value of the slope when x is close to 4. How does the slope behave as x gets closer and closer to 4? Repeat for the point $(-4, 16)$.

P2.2 Which description matches the graph below?

(a) Casey walked from home to the park. Starting off slowly, Casey's pace gradually increased. At the park Casey turned around and walked slowly back home.

(b) From home Casey took a bike ride heading east going up a steep hill. After a while the slope eased off. At the top Casey raced down the other side.

(c) From home Casey went for a jog. At the end of the street Casey bumped into a friend and slowed down. After Casey's friend left, Casey walked quickly back home.



P2.3 The first few terms of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, \ldots$, and each successive term is the sum of the two previous terms. Make a table of differences for this sequence. What patterns do you notice in the first differences? The second differences? And so on?

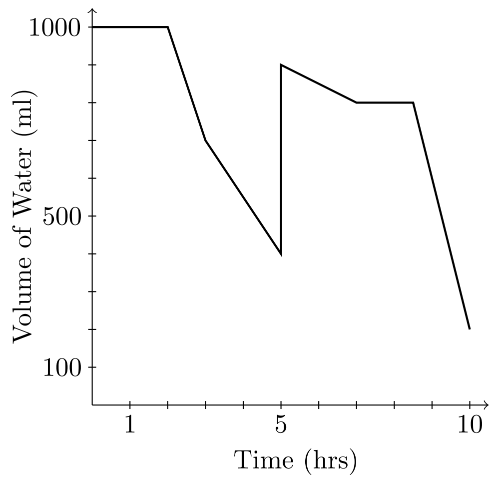
P2.4 The half-life of ibuprofen in the bloodstream is about 2 hours. This means that after 2 hours of being consumed, only 100 mg of the 200 mg of ibuprofen taken will remain in the bloodstream. After another 2 hours (4 hours total) only 50 mg will remain in the bloodstream. A patient is prescribed 200 mg of ibuprofen to be taken every 4 hours. Fill in the following table, which records the amount of ibuprofen in the patient's body hours after the initial dose is consumed. What is the long-term behavior of the concentration?

Elapsed time in hours	0	2	4	6	8	10	12	14	16
Mg of ibuprofen in the body	200	100	250						

P2.5 The zeros, or zeroes, of the function Q are -4 and 5. Find the zeros of the given functions.

(a) $f(x) = Q(4x)$ **(b)** $t(x) = Q(x - 3)$ **(c)** $j(x) = 3Q(x + 2)$

P2.6 The graph below shows the volume of water in Jo's water bottle over time. Use the graph to tell the story of Jo's water consumption.



P2.7 Suppose a herd of goats currently number 25. Consider two possible rates of growth for the herd and compare the size of the herd over time for these two growth scenarios.

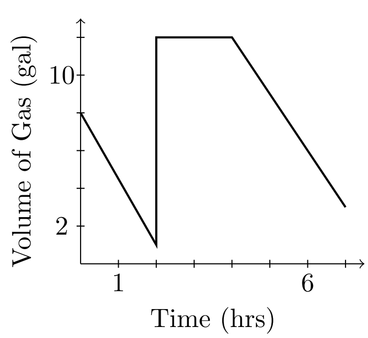
(a) An increase of 10 goats per year.

(b) An increase of 20 % per year.

P2.8 Suppose we know the following about the function $f(x)$. The domain is $-10 \leq x \leq 15$, the range is $-20 \leq y \leq 5$, the x-intercepts are $(-1, 0)$ and $(7, 0)$, the y-intercept is at $y = -10$. For each function, identity the domain, the range, and x-intercepts and the y-intercept, if possible. If there is not enough information to identify any feature, explain why this is the case.

(a) $g(x) = -2f(2x)$ **(b)** $m(x) = f(x + 5) - 1$

P2.9 The graph below shows the volume of gas in Cam's car while driving to New York City. Use the graph to tell the story of Cam's gas use during the trip.



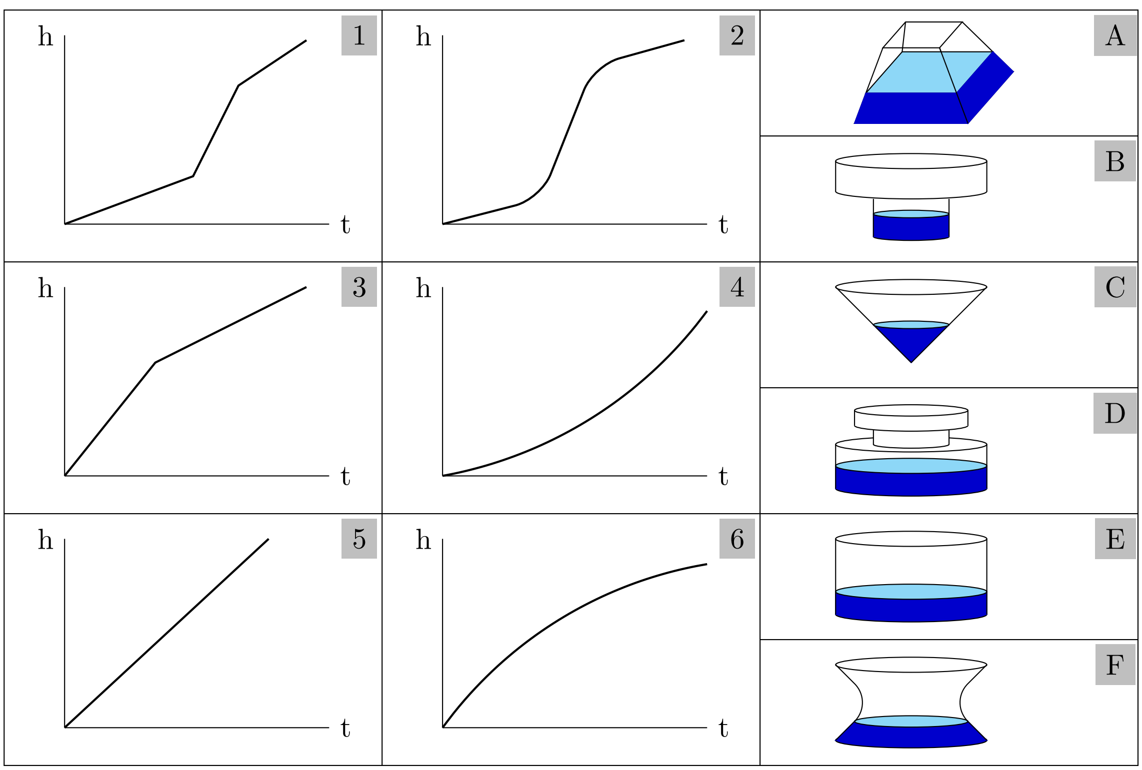
P2.10 Sketch a graph of distance (in miles) from Exeter versus time (in hours) for a car trip you take on a Sunday afternoon from PEA to Deerfield Academy, which is about 120 miles away. Some of the time you will be driving in towns, most of the time you will be on highways, and you will want to make a stop to eat some lunch about midway through your trip. Next, sketch a graph of the rate-of-change function that corresponds to your distance vs. time graph.

P2.11 Sketch the graph of a function that has a positive and increasing rate of change.

P2.12 A driver was overheard saying, "My trip to New York City was made at 80 kilometers per hour." Do you think the driver was referring to an instantaneous speed or an average speed? What is the difference between these two concepts?

P2.13 (Continuation) Let $R(t)$ denote the speed of the car after t hours of driving. Assuming that the trip to New York City took exactly five hours, draw a careful graph of a plausible speed function R. It is customary to use the horizontal axis for t and the vertical axis for R. Each point on your graph represents information about the trip; be ready to explain the story behind your graph. In particular, the graph should display reasonable maximum and minimum speeds.

P2.14 Water is flowing into each container at a constant rate. As the volume increases the height of the water also increases. Match each graph of height versus time with its container.



P2.15 For which of the following functions can it be said $f(a+b) = f(a) + f(b)$ for all a and b in the domain of the given function?

(a) $f(x) = -\frac{x}{2}$, **(b)** $f(x) = x^2$, **(c)** $f(x) = \frac{1}{x}$.

P2.16 The function p defined by $p(t) = 3960(1.02)^t$ describes the population of Dilcue, North Dakota, t years after it was founded.

(a) Find the founding population.

(b) At what rate was the population growing ten years after the founding?

(c) At what annual rate has the population of Dilcue been growing?

P2.17 Without a calculator and on the same set of axes, graph $f(x) = x$ and $y = \frac{1}{f(x)}$.

Prelaboratory 2: Differences and How Polynomials Change

PL2.1 Using a spreadsheet (Google Sheets, Excel, etc.) and the template of Table 1 (shown below), make a table of values where the first column is $x = 0, 1, 2....$ Then, fill in the second column $y = x^2$.

PL2.2 Fill in the third column Diff_1 of Table 1 with the differences between the successive y-values in column two. Is there a pattern to the column Diff_1? Do the values in this column fit a linear function? Explain.

PL2.3 Fill in the fourth column Diff_2 that shows the differences of the differences, also known as the second differences. You should get the same constant value for each row in this column. How does this relate to the linear function you found?

Table 1: Differences

x	$y = x^2$	Diff_1	Diff_2
0	0	1	2
1	1	3	
2	4		
3			
4			
5			-
6		-	-

PL2.4 Repeat the steps of Problem 1 for the function $y = x^2 + x + 1$ by filling in column 1 with $x = 0, 1, 2, ...$, column 2 with the values of $y = x^2 + x + 1$, column 3 with the first differences of the successive y-values in column 2, and column 4 with the second differences by calculating the differences of the successive values in column 3.

Lab 2: Differences and How Polynomials Change

L2.1 Using the three coefficients a, b, c assigned to you by your teacher from Table 2, for $y = ax^2 + bx + c$, fill in the y-values, Diff_1, and Diff_2 in Table 2 similar to what was done in the Prelab problems. How does the constant in your Diff_2 column relate to the values in the Diff_1 column? Along with your classmates, enter the constant value from your results in the column for Diff_2 in Table 2.

Table 2: Class Data

Group	a	b	c	Diff_2
	1	1	0	
	-1	1	1	
	2	2	1	
	-2	1	2	
	3	-1	0	
	3	-1	2	
	-3	3	1	
	4	1	1	
	4	1	3	
	-4	2	0	
	5	1	1	
	π	1	1	

L2.2 How does the number in Diff_2 relate to a, the coefficient of x^2? Does the same relation hold for negative values of a? Do the values of b and c have any effect on the value of Diff_2?

L2.3 Make a table similar to Table 1 in the Prelab for the function $y = x^3$. Include as many columns of differences as necessary until you get a constant difference like you did for the quadratic. How many differences are needed to get a constant difference in this cubic polynomial?

L2.4 Using the four coefficients assigned to you from Table 3, for $y = ax^3 + bx^2 + cx + d$, make a table similar to what you created for the previous problems. When you get to the column that has a constant difference, add that value to Table 3 and share your results with the class.

Table 3: Class Data

Group	a	b	c	d	Diff_3
	1	1	0	0	
	-1	1	1	1	
	2	0	0	0	
	2	2	1	0	
	-2	1	2	1	
	3	-1	0	0	
	3	-1	2	1	
	4	1	3	0	
	-4	2	0	2	
	5	1	1	1	
	6	1	0	1	
	π	1	-1	2	

L2.5 How does the number in the constant difference column relate to the coefficients a, b, c, or d?

L2.6 What type of function is represented by the Diff_2 column for your cubic polynomial? By extension, what function is represented by the values in your Diff_1 column?

L2.7 What do you think will happen if you calculate differences for a fourth-degree polynomial? Test your conjecture on a fourth-degree polynomial of your choosing.

L2.8 If you were told it took 5 difference columns to get to a constant difference, what do you know about the function you started with?

L2.9 You have now worked with quadratic, cubic, and quartic (4th degree) polynomials and the associated difference tables. In general, how many differences are needed to get to a constant difference for a polynomial of degree 2? 3? 4? n? Also, consider what degree functions are represented by the other difference columns. (You have a column of constant differences. Is there a linear column? Is there a quadratic column?) How does the constant difference column relate to the coefficients of the polynomial? What patterns do you notice? Write a paragraph summarizing your observations about differences for polynomial functions. Include any tables you feel may be helpful to your explanations.

P3.1 Using a strategy like the one you used in Lab 2, create a table of differences for the function $y = 2^x$. You should see that the difference of consecutive y-values never becomes 0, but does behave in a predictable way. Repeat for $y = 3^x$ and $y = 1.2^x$. Describe how differences for an exponential function behave compared to differences for a polynomial function.

P3.2 The equation $V(t) = 8000(0.95)^t$ models the volume in cubic centimeters of a shrinking balloon that is losing 5 percent of its helium each day.

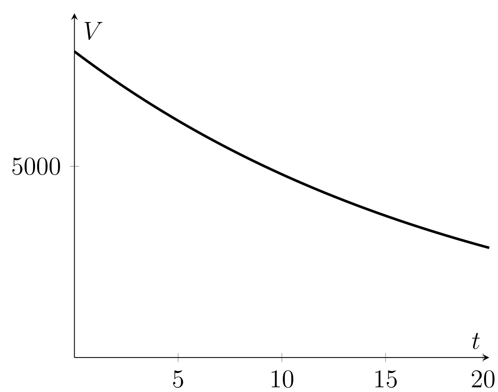
(a) Calculate $V(0)$. What does the value tell you about the graph? What does the value tell you about the balloon?

(b) Calculate $V(10)$. What does the value tell you about the graph? What does the value tell you about the balloon?

(c) Find t so that $V(t) = 5000$. Describe the balloon at this moment.

(d) Calculate $\dfrac{V(14) - V(12)}{14 - 12}$. What does the value of the fraction tell you about the graph? What does this value tell you about the balloon?

(e) Calculate $\dfrac{V(14) - V(13)}{14 - 13}$. What does this value tell you about the balloon?



P3.3 Match each of the following descriptions with its distance vs. time graph.

(a) Rory ran from home to the bus stop and waited. Rory realized the bus had already left so Rory walked home.

(b) Opposite Rory's home is a hill. Rory climbed slowly up the hill, walked across the top, and then ran quickly down the other side.

(c) Rory skateboarded from home, gradually building up speed. Rory slowed down to avoid some rough ground, but then speeded up again.

(d) Rory walked slowly along the road and stopped to check the time. Seeing how late it was Rory then started running.

(e) Rory left home for a run. Rory soon got tired and gradually came to a stop!

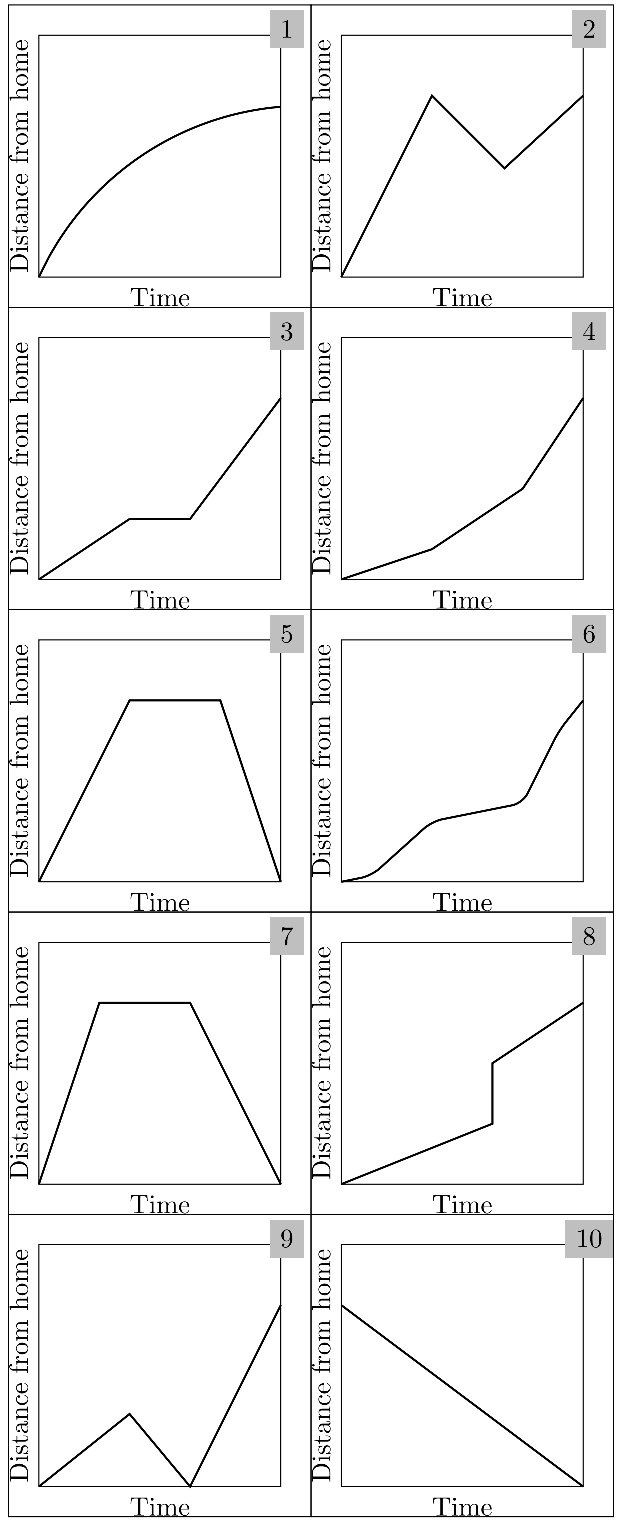
(f) Rory walked to the store at the end of the street, bought a newspaper and then ran all the way back.

(g) Rory went out for a walk with some friends. After noticing Rory's wallet was at home, Rory ran home to get it and then had to run to catch up with the others.

(h) This graph is just plain wrong. How can Rory be in two places at once?

(i) After the party, Rory walked slowly all the way home.

(j) Make up your own story!



P3.4 When logarithms are calculated using e as the base, they are called *natural*, and written $y = \ln(x)$ instead of $y = \log_e(x)$. Graph $y = e^x$ and $y = \ln(x)$ on the same set of axes. Recall that exponential functions and logarithmic functions with the same base are inverse functions.

(a) How are the graphs of these two functions related? Compare the domains and ranges of the functions.

(b) What are the axis intercepts for the two curves? Estimate the slope of the tangent lines to the curves at the intercepts. The slope of the tangent line to a curve at a point *is* the *slope of the curve at that point*.

P3.5 Sketch the graph of a function that has the property that its rate of change is linear. How does your graph compare with the graphs of your classmates?

P3.6 Sketch the graph of a function whose rate of change is:

(a) always positive, **(b)** always negative.

P3.7 For which of the following functions can it be said $f(a + b) = f(a) + f(b)$ for all a and b in the domain of the given function?

(a) $f(x) = 3x$, **(b)** $f(x) = -x + 5$, **(c)** $f(x) = \sqrt{x}$, **(d)** $f(x) = 2^x$.

P3.8 On the graph of $y = f(x)$, it is given that $(-2, 5)$ is the highest point, $(2, -7)$ is the lowest point, and $x = -4$, $x = 1$ and $x = 3$ are the x-intercepts. Find the highest and lowest points on the graph as well as the x-intercepts of the curves.

(a) $y = f(x - 5) + 8$, **(b)** $y = 3f(2x)$.

P3.9 The point $A(0, 1)$ is on the graph of $f(x) = 2^x$. If $B(x, 2^x)$ is any other point on the graph of $f(x) = 2^x$, the the fraction $\dfrac{2^x - 1}{x - 0}$ represents the slope between the two points A and B on the graph. Find the value of the slope when x is close to 0. How does the slope behave as x gets closer and closer to 0?

P3.10 Given a function f, each solution of the equation $f(x) = 0$ is called a *zero* of f. Without using a calculator, find the zeros of each function.

(a) $s(x) = \sin(3x)$ **(b)** $L(x) = \log_5(x - 3)$

P3.11 Using the distance vs. time graph in #P2.2, sketch a graph of Casey's rate of change.

P3.12 The point $(0,1)$ is the y-intercept on the graph of $y = b^x$ for $b > 0$. The expression $\dfrac{b^x - 1}{x - 0}$ approximates the slope of $y = b^x$ at its y-intercept, when x is close to 0. Fill in the missing entries in the following table.

b	Slope of $y = b^x$ at its y-intercept
0.075	
0.5	
1.5	
2	
3	
5	
8	
15	

P3.13 (Continuation) Make a scatterplot with values of b on the horizontal axis and the value of the slope of $y = b^x$ on the vertical axis. The shape of the scatterplot suggests a simple relationship between the slope and the value of b. What familiar function describes this relationship?

P3.14 We have now discussed problems with differences and slopes for various functions. These problems are actually about rates of change, specifically how a function changes with respect to its independent variable. The change of a function can be thought of as another function, which for now, we will call the *rate-of-change function*. Use your experience and intuition to respond to the following:

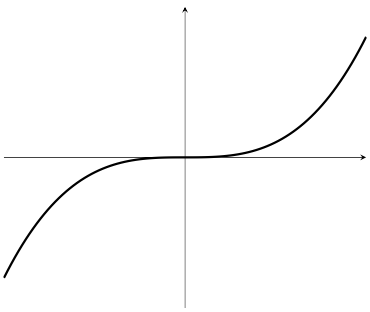
(a) Describe the rate of change for $y = 3x+1$. Describe the rate-of-change function for any linear function.

(b) Describe the rate-of-change function for a parabola.

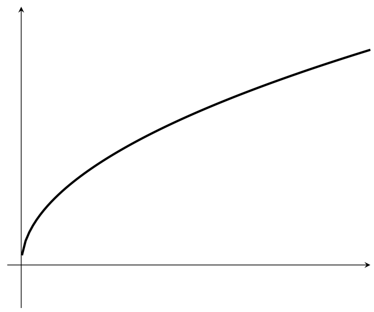
(c) Describe the rate-of-change function for an exponential function.

Prelaboratory 3: Approximating Instantaneous Rate of Change

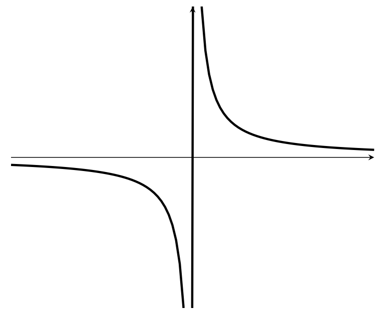
PL3.1 On the graphs below, sketch the rate of change function for each of the functions shown.



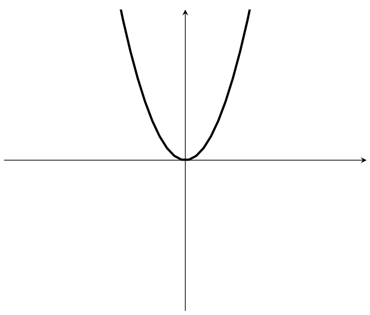
(a) $y = x^3$



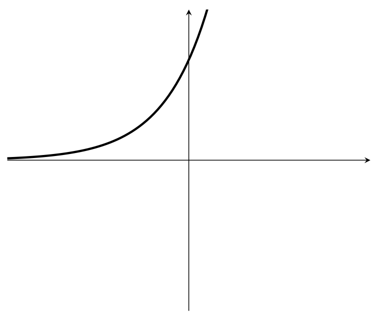
(b) $y = \sqrt{x}$



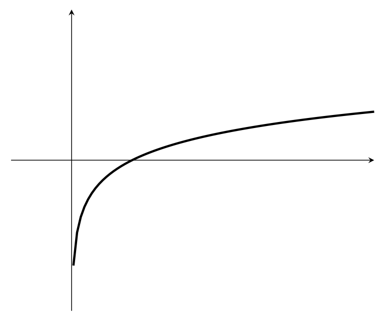
(c) $y = \dfrac{1}{x}$



(d) $y = x^2$

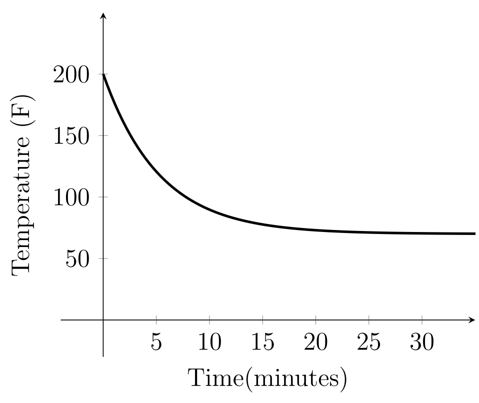


(e) $y = e^x$



(f) $y = \ln x$

PL3.2 The following graph illustrates the temperature of a fresh cup of coffee as it cools. Sketch a graph of the rate-of-change curve of this *cooling curve*.



Laboratory 3: Approximating Instantaneous Rate of Change

L3.1 In the prelab assignment for this lab, you sketched rate of change graphs. The in-class portion of this lab focuses on refining the estimations you have already made. Use a graphing app to view the function $f(x) = \sin(x)$ over an appropriate domain, where x is in radians. (You many change your mind about what is *appropriate* as you work through this lab.)

L3.2 Create a new function $g(x)$ where

$$g(x) = \frac{f(x + 0.001) - f(x)}{0.001}.$$

Notice that for a given x-value, $g(x)$ describes the average rate-of-change of $f(x)$ on the interval $[x, x + 0.001]$. Obtain a graph of $g(x)$. This average rate-of-change function is an approximation for the instantaneous rate-of-change function, which we call the *derivative*. This average rate-of-change function can help us both understand and find an equation for the derivative.

L3.3 (Continuation) Have you seen a function that looks like $g(x)$ before? Make a guess as to which well-known function is the derivative of $\sin(x)$. Verify or debunk your guess by graphing it on the same axes as the approximation function $g(x)$.

L3.4 Repeat steps 1 through 3 above for each function below:

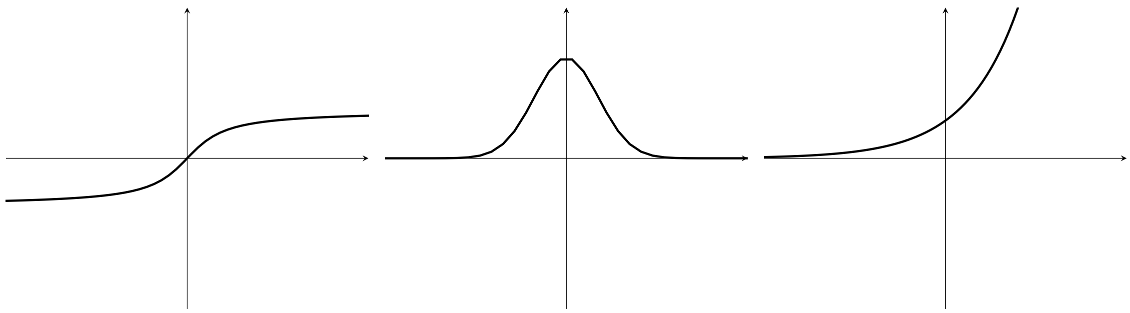
(a) $y = x^2$ **(b)** $y = x^3$ **(c)** $y = \cos(x)$ **(d)** $y = e^x$

(e) $y = \dfrac{1}{x}$ **(f)** $y = \dfrac{1}{x^2}$ **(g)** $y = \ln(x)$ **(h)** $y = \sqrt{x}$

Make sure to keep a record of your results including a sketch of each function and its derivative along with their equations.

L3.5 Write a brief report summarizing what you have learned in this lab. Include a table listing all of the function equations from the prelab, your guess for those derivatives, and what you discovered through the lab. Note any patterns you see, including connections to Lab 2. Also, note any generalizations you might be willing to make and any result that surprised you.

P4.1 Using a technique similar to what you used in Lab 3, sketch the rate of change function for each of the following.



(a) $y = \arctan(x)$ **(b)** $y = e^{-x^2}$ **(c)** $y = 2^x$

P4.2 Use the results of Lab 3 to find the slope of the function $f(x) = x^2$ at $x = -\dfrac{3}{2}$, and then write the equation of the line tangent to $f(x) = x^2$ at $x = -\dfrac{3}{2}$. Use a graphing app to graph $y = f(x)$ and this tangent line on the same axes.

P4.3 Use the results of Lab 3 to find three points on the graph of $y = \sin x$ where the slope is equal to $\dfrac{1}{2}$.

P4.4 After being dropped from the top of a tall building, the height of an object is described by $h(t) = 400 - 16t^2$, where $h(t)$ is measured in feet and t is measured in seconds.

(a) How many seconds did it take for the object to reach the ground? Use a graphing app to obtain a graph of height vs. time in a suitable window.

(b) What is the height of the object when $t = 2$? What is the height of the object when $t = 4$? Use algebra to obtain an equation for the line that goes through the two points you just calculated. Add your line to the graph of height vs. time. A line that is determined by two points on a curve is known as a *secant line*.

(c) What is the average rate at which the height of the object is changing between 2 and 4 seconds after it is dropped? How does this rate relate to the slope of the secant line?

P4.5 (Continuation)

- **(a)** How fast is the height of the object changing at the instant 2 seconds after being dropped?
- **(b)** Explain why this rate of change is given by the *difference quotient* $\dfrac{h(t)-h(2)}{t-2}$, where t is a time very close to 2 seconds.
- **(c)** Verify that the closer t is to 2, the closer the rate of change is to -64.

We can summarize this situation by writing $\lim\limits_{t\to 2}\dfrac{h(t)-h(2)}{t-2}=-64$. The notation $\lim\limits_{t\to 2}$ is read as "the limit as t approaches 2". This limit is known as the instantaneous rate of change when $t=2$.

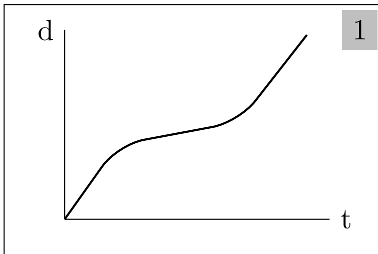
P4.6 (Continuation)

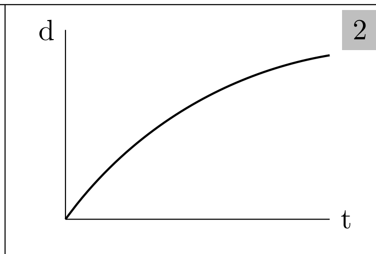
- **(a)** What is an equation for the tangent line to the curve at the point with $t=2$?
- **(b)** How is the slope of the tangent related to the instantaneous rate of change at that point?
- **(c)** Use a graphing app to graph the curve $h(t)=400-16t^2$ together with the tangent line at $t=2$.
- **(d)** Gradually zoom in on the point with $t=2$. What do you notice about the curve and the tangent line as you zoom in? Because the curve and the tangent line are indistinguishable on a small interval containing $t=2$, this curve is *locally linear* at $t=2$.

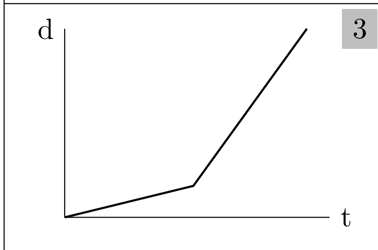
P4.7 (Continuation) Suppose we now drop the object from a taller building, so that the initial height is 600 ft, thus $h(t)=600-16t^2$.

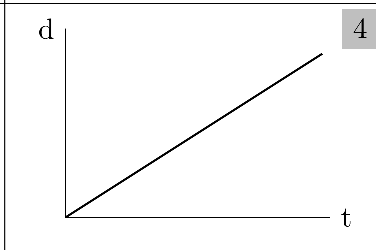
- **(a)** What is the height of the object at time $t=2$?
- **(b)** Find the instantaneous rate of change of the height at $t=2$.
- **(c)** Find the equation of the tangent line to the curve at the point where $t=2$.
- **(d)** How do your answers compare with your results for the problem with initial height 400 ft?

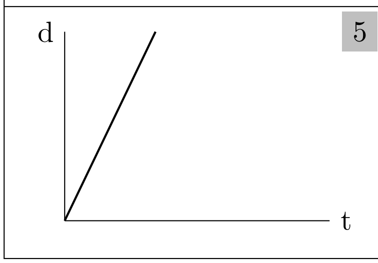
P4.8 Match the graphs with the descriptions:

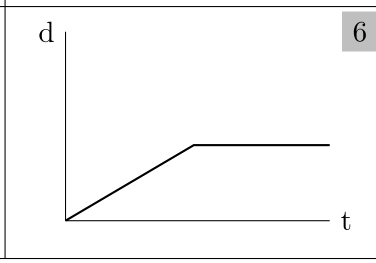










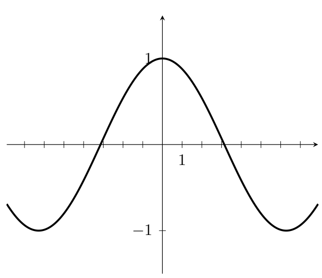


- **A** Moving at a steady constant pace.
- **B** Moving at a fast pace, changing gradually to a slower pace.
- **C** Moving at a fast steady pace.
- **D** Moving fast, then slowing slightly, then going faster again.
- **E** Moving at a steady pace and then stops for a period of time.
- **F** Moving at a slow pace and then rapidly increases.

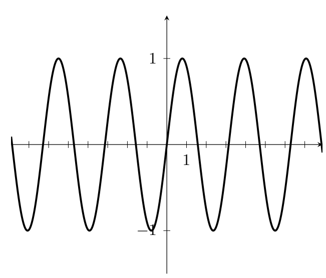
P4.9 Use the results of Lab 3 to find the slope of the function $f(x) = \frac{1}{x}$ at $x = \frac{5}{2}$, and then write the equation of the line tangent to $f(x) = \frac{1}{x}$ at $x = \frac{5}{2}$. Use a graphing app to graph $y = f(x)$ and its tangent line on the same axes.

P4.10 Decide which of the following equations are graphed below:

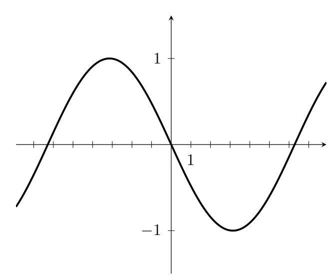
$y = \sin(2x)$, $y = \cos(0.5x)$, $y = \sin(-0.5x)$, $y = \cos(-2x)$ $y = \sin(3x)$



(a)



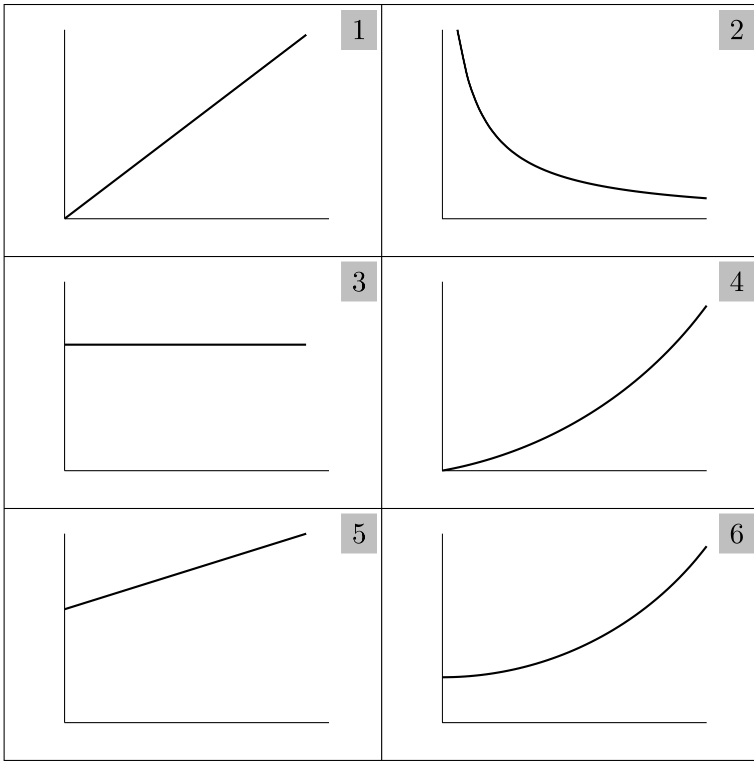
(b)



(c)

P4.11 Use the difference quotient technique of Lab 3 to obtain a graph of an approximation to the derivative of $f(x) = \tan x$. Now, graph $g(x) = \dfrac{1}{\cos^2(x)} = \sec^2(x)$. What can you conclude from this?

P4.12 Match the graphs with the descriptions:



A. The cost of hiring an electrician per hour, including a fixed call out fee.

B. The connection between the length and width of a rectangle of fixed area.

C. Speed against time for a car traveling at constant speed.

D. The area of a circle as the radius increases.

E. The width of a square as the length of the square increases.

F. The 1954 population increased slowly at first, then increased more quickly.

P4.13 You have seen that the expression $\left(1+\dfrac{1}{n}\right)^n$ gets closer and closer to "e" as n approaches ∞, which is expressed formally as $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. The limit notation of this equation implies that the difference between e and $\left(1+\dfrac{1}{n}\right)^n$ can be made small by choosing a large value of n. How large does n need to be so that the difference between $\left(1+\dfrac{1}{n}\right)^n$ and e is less than 0.01? 0.0001? 0.000001?

P4.14 Working in radians, evaluate $\dfrac{\sin(0+\Delta t)-\sin(0)}{\Delta t}$ for $\Delta t = 0.1$, $\Delta t = -0.1$, $\Delta t = 0.01$, and $\Delta t = -0.01$. Based on your results, what would you say is the value of $\displaystyle\lim_{\Delta t\to 0}\frac{\sin(0+\Delta t)-\sin(0)}{\Delta t}$?

P4.15 Using the graph of $y = f(x)$ below, evaluate each of the following, and be ready to share your observations with your classmates.

(a) $\lim_{x\to 3} f(x)$

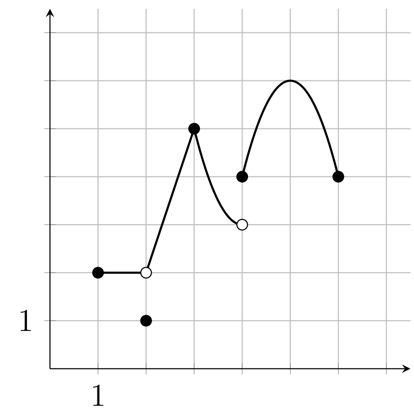
(b) $f(3)$

(c) $\lim_{x\to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x\to 4} f(x)$

(f) $f(4)$



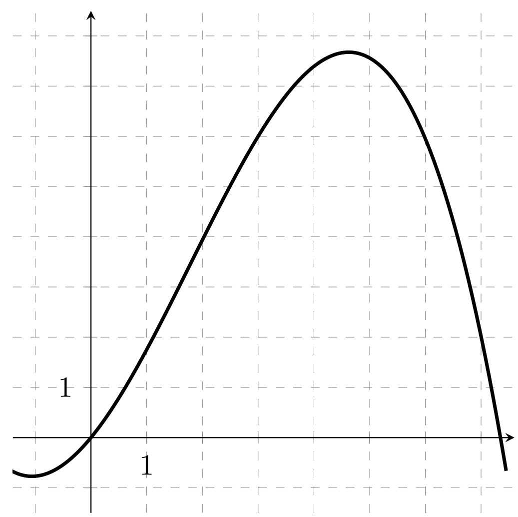
P4.16 On the graph $y = f(x)$ shown below, draw lines whose slopes are

(a) $\dfrac{f(7) - f(3)}{7 - 3}$

(b) $\lim_{h\to 0} \dfrac{f(6+h) - f(6)}{h}$

(c) $\dfrac{f(7) - f(0)}{7 - 0}$

(d) $\lim_{h\to 0} \dfrac{f(h)}{h}$



P4.17 (Continuation) On the graph $y = f(x)$ above, mark points where the x-coordinate has the following properties (a different point for each equation):

(a) $\dfrac{f(x) - f(2)}{x - 2} = \dfrac{1}{2}$,

(b) $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = -1$,

(c) $\dfrac{f(x)}{x} = \dfrac{1}{2}$,

(d) $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = 0$.

P4.18 (Continuation) On a separate system of axes, graph the derivative function f'.

P4.19 What is the value of each limit? How does each value relate to the graphs of $f(x) = \tan^{-1}(x)$ and $g(x) = \dfrac{x^2 - 1}{x^2 + 1}$?

(a) $\lim\limits_{x\to\infty} \tan^{-1} x$, **(b)** $\lim\limits_{x\to-\infty} \tan^{-1} x$, **(c)** $\lim\limits_{x\to\infty} \dfrac{x^2 - 1}{x^2 + 1}$, **(d)** $\lim\limits_{x\to-\infty} \dfrac{x^2 - 1}{x^2 + 1}$.

P4.20 Use the results of Lab 3 to find a point on the graph of $y = \ln(x)$ where the slope is equal to **(a)** 1 **(b)** 2 **(c)** $\dfrac{1}{3}$

P4.21 Choose a value of b, $b > 0$ and $b \neq 1$, and graph the function $y = \dfrac{b^x - 5}{b^x + 3}$.

(a) What happens to the y-values as x increases without bound, $x \to \infty$?

(b) What happens to the y-values as x decreases without bound, $x \to -\infty$?

Prelaboratory 4: Linear Approximation

PL4.1 Graph the lines with the following equations. What is the same about all the graphs? What is different?

(a) $y = 3(x-1)+2$

(b) $y = -2(x-1)+2$

(c) $y = \frac{1}{2}(x-1)+2$

PL4.2 Find an equation for the line with a slope of 1.5 containing the point $(-2, 3)$. Use the *point-slope formula* in your work.

PL4.3 What is the derivative of $y = \sqrt{x}$ at the point with $x = 4$? What is the slope of the tangent to the curve at this point? Write the equation of the tangent line at this point.

PL4.4 The tangent line in #PL4.3 can be used to approximate the value of $y = \sqrt{x}$ at points that would otherwise require a calculator. Graph $y = \sqrt{x}$ and the tangent line at $x = 4$. Estimate the error in the tangent line approximation compared with the value on the curve at $x = 3$.

Laboratory 4: Linear Approximation

L4.1 Linear approximation - Part 1

(a) Graph the function $y = x^2$

(b) What is the value of the derivative at the point with $x = 1$? Calculate the equation of the tangent line at this point.

(c) The tangent line provides a linear approximation for the curve near the point with $x = 1$. Calculate the value of this linear approximation at $x = 2$. Is the linear approximation an overestimate or an underestimate of the actual value on the curve $y = x^2$? What is the *error* of the linear approximation? Use the error to calculate the *percent error* of the linear approximation.

L4.2 Linear approximation - Part 2

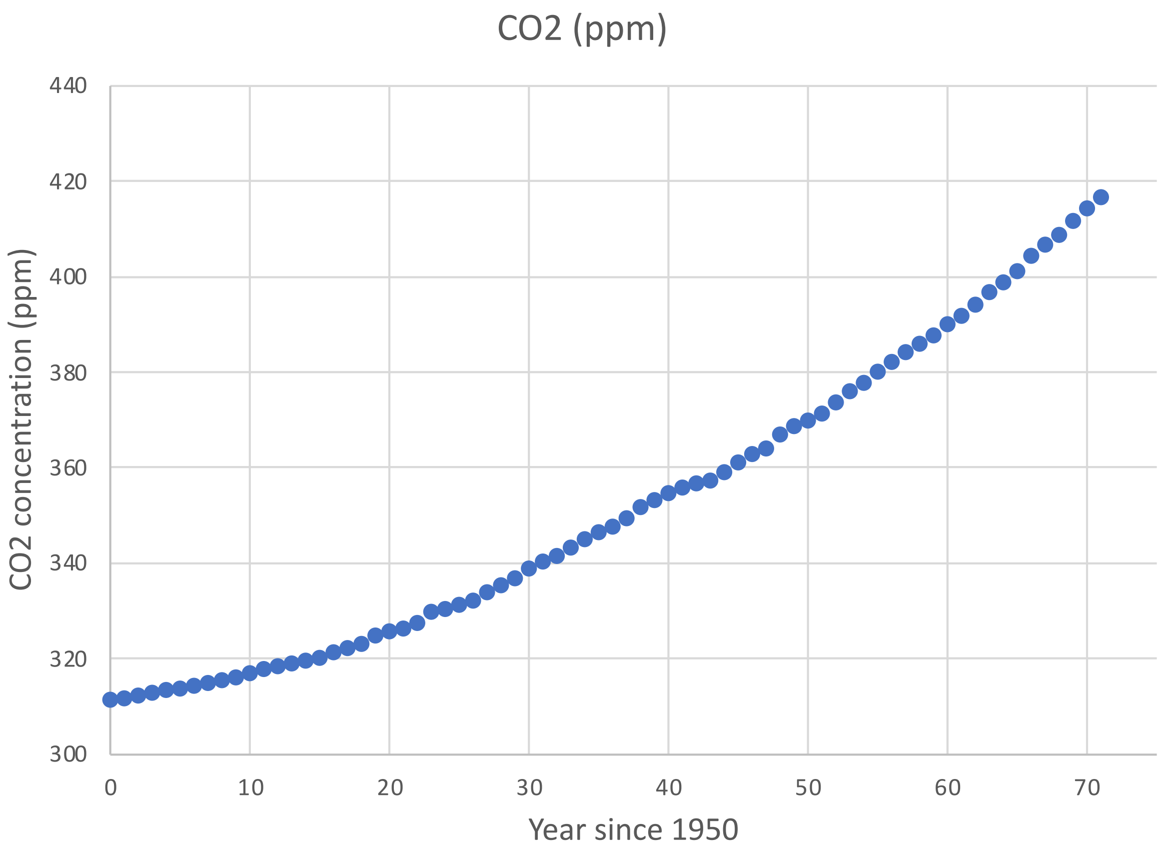
(a) Graph the function $y = \sin(x)$

(b) What is the value of the derivative at the point with $x = 0$? Calculate the equation of the tangent line at this point, which is the linear approximation *centered* at this point.

(c) Calculate the value of this linear approximation at $x = 1$. Is the linear approximation an overestimate or an underestimate of the actual value on the curve $y = \sin(x)$? What is the percent error of the estimate?

L4.3 How can you determine, based on the shape of the graph of a function, whether a linear approximation is an overestimate or an underestimate?

L4.4 Linear Projection: The following graph is a plot of the data for the concentration (parts per million, or ppm) of atmospheric CO_2 versus years since 1950. For example, the point (71, 416.45) indicates that in 2021 the CO_2 concentration in the atmosphere was 416.45 ppm (which is an average of the CO_2 concentration for the year 2021). Two possible projections for future CO_2 concentration are to use the curve fit to the historic data over the past 70 years or to use a line tangent to that curve at a recent point.



(a) Use the data for the last 10 years to calculate the equation of a line that fits the trend in the data for this decade. Remember that the year represents years since 1950.

Year	62	63	64	65	66	67	68	69	70	71
CO_2	394.1	396.7	398.9	401.0	404.4	406.8	408.7	411.7	414.2	416.5

(b) The equation for the curve that fits the historic data since 1950 is $y = 30.2(1.022)^x + 280$. Confirm that this exponential function and the line from the previous part give approximatly the same CO_2 concentration for $x = 71$.

(c) Compare the linear projection with the exponential projection at $x = 100$ (which is year 2050). Do the same for the year 2100. In each year, 2050 and 2100, what percent lower is the linear projection?

L4.5 Local Linearity

(a) Using technology to graph the function $y = x^2$, zoom in on the curve centered at the point $(1, 1)$. Notice that as you zoom in, this smooth curve appears straighter. This property, called *local linearity*, is fundamental to many concepts in calculus.

(b) Using technology, graph the function $y = \sin(x)$ and the tangent to the curve at the point $(0, 0)$. What do you notice about the curve and the tangent line as you zoom in on the graph centered at $(0, 0)$?

(c) Why is a tangent line a reasonable approximation for a curve near the point of tangency?

L4.6 Graph the following function using technology:

$$y = \begin{cases} x^2 & \text{if} \quad x \leq 2, \\ -x + 6 & \text{if} \quad x > 2. \end{cases}$$

(a) Gradually zoom in around the point $(2, 4)$, which is where the definition of the function changes. Is the function locally linear at $(2, 4)$? In other words, do you see one line or two lines when you zoom in?

(b) What are the slopes of the two lines that meet at $(2, 4)$ when you zoom in on that point? Is there a unique linear approximation centered at $(2, 4)$? Is there a unique tangent line at $(2, 4)$? This sharp point is called a *cusp*.

L4.7 Summarize in a paragraph, with accompanying pictures, what you have learned in this lab about linear approximation. Be sure to explain what a linear approximation is, how it relates to a linear projection, how to calculate a linear approximation, what determines if it is an overestimate or an underestimate, and how linear approximations are connected to local linearity.

P5.1 Consider the function $f(x) = x^2$.

(a) Find the slope of the line between the points $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$ for $\Delta x = 0.5$, $\Delta x = 0.1$, $\Delta x = 0.05$, and $\Delta x = 0.01$. What happens to the slope as Δx approaches zero? Use a limit to write this result.

(b) Obtain an expression for $\dfrac{f(3 + \Delta x) - f(3)}{\Delta x}$ by substitution using $f(x) = x^2$, then simplify this expression using algebra. Use the result to evaluate the limit expression $\lim\limits_{\Delta x \to 0} \dfrac{f(3 + \Delta x) - f(3)}{\Delta x}$.

P5.2 For $f(x) = x^2$, simplify the expression $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ and use the result to evaluate $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. What does this limit tell you about the derivative of f?

P5.3 For $f(x) = x^2 - 3x$, simplify the expression $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ and use the result to evaluate $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. What does this tell you about the derivative of f?

P5.4 In the quadratic functions of the three previous problems, why did you need to simplify the expressions before evaluating the limits?

P5.5 Consider the limit $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

(a) Use algebra to evaluate $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ by first simplifying the ratio $\dfrac{x^2 - 1}{x - 1}$.

(b) Compare the graphs of $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$, and the domains of each function.

P5.6 In our previous work we have often used a difference quotient to represent the rate of change of a function. Write a paragraph explaining your understanding of the limit of a difference quotient as an expression of the derivative of $f(x)$:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The expression on the right side of the equation above is known as the *limit definition of the derivative.*

P5.7 A *linear function* has the form $L(x) = mx + b$ where m and b are constants (that is, the values of m and b do not depend on the value of x). What is the derivative $L'(x)$? Use algebra and the limit definition of the derivative to show how to go step-by-step from $L(x)$ to $L'(x)$.

P5.8 You have discovered the derivatives for the following functions. Fill in the table with the derivatives.

$f(x)$	$f'(x)$
x^2	
x^3	
$\sin x$	
$\cos x$	
e^x	
$\dfrac{1}{x}$	
$\dfrac{1}{x^2}$	
$\ln x$	
$\sqrt{x}$	

P5.9 Consider the absolute value function $y = |x|$, which can be defined as

$$y = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

(a) When a function is piecewise defined, we need to consider the derivative of each piece separately. What is the derivative of each "half" of the function? What is the derivative at $x = 0$? How do you know?

(b) To gain insight into part (a), zoom in on the graph of $y = |x|$ around $(0, 0)$. Do you see local linearity? Explain how the picture relates to the derivative or lack of a derivative for this function.

P5.10 **(a)** Draw the graph of $y = \sin x$ for x in radians, $-2\pi \leq x \leq 2\pi$. Now find the slope of the curve at the origin. Is your answer consistent with what you know is the derivative of $y = \sin x$ at $x = 0$? Explain.

(b) Draw the graph of $y = \sin x$ for x in degrees, $-360 \leq x \leq 360$. Now find the slope of the curve at the origin. Is your answer consistent with what you know is the derivative of $y = \sin x$ at $x = 0$? Explain.

(c) Why do we generally use radians with trig functions in calculus?

P5.11 (Continuation) Working in radians, evaluate $\lim_{x \to 0} \dfrac{\sin x}{x}$. Interpret your answer.

P5.12 Consider the piecewise-defined function $f(x) = \begin{cases} -x+4 & \text{if} \quad x < 0, \\ x^2+4 & \text{if} \quad x \geq 0. \end{cases}$

(a) Why is this function not differentiable everywhere? When a function fails to have a derivative at a point, it is said to be *nondifferentiable* at that point.

(b) Adjust the linear part of f to make f differentiable everywhere.

(c) Now, instead, let the quadratic part of the original f be $x^2 + bx + 4$. What value of b makes f differentiable everywhere?

P5.13 Why does $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ represent the same value as $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$? Draw a picture to justify your answer.

P5.14 This function indicates the tax $T(x)$ for each nonnegative taxable income x.

$$T(x) = \begin{cases} 0.10x & \text{for} \quad 0 \leq x \leq 18\,150 \\ 1815 + 0.15(x - 18\,150) & \text{for} \quad 18\,150 < x \leq 73\,800 \\ 10\,162.50 + 0.25(x - 73\,800) & \text{for} \quad 73\,800 < x \leq 148\,850 \\ 28\,925 + 0.28(x - 148\,850) & \text{for} \quad 148\,850 < x \leq 226\,850 \\ 50\,765 + 0.33(x - 226\,850) & \text{for} \quad 226\,850 < x \leq 405\,100 \\ 109\,587.50 + 0.35(x - 405\,100) & \text{for} \quad 405\,100 < x \leq 457\,600 \\ 127\,962.50 + 0.396(x - 457\,600) & \text{for} \quad 457\,600 < x \end{cases}$$

(a) What is the tax for a couple whose taxable income is \$50 000? \$73 800? \$100 000? \$300 000?

(b) What is the meaning of "I'm in the 25% tax bracket"? What about the 33% bracket? Why does each piece (except for the first) of T begin with a number (1815 for example)?

(c) Open the Desmos file provided by your teacher which contains a definition and graph of the function.

(d) Explain why $T'(x)$ makes sense for all but six positive values of x. For these six values, why is T nondiferentiable?

(e) Draw a rough sketch of the derivative function T'. How many distinct values are there in the range of T'?

P5.15 Find the average rate of change for $y = x^2$ between $x = 0$ and $x = 2$. Is there a point with the x-value in the interval $(0, 2)$ where the instantaneous rate of change is the same as the average rate of change over the interval? Explain and find the point. Verify your calculations with a graph.

P5.16 (Continuation) How would you explain your answer to the previous problem with slopes and lines?

P5.17 Find an approximate value for $F(2.3)$ given only the information $F(2.0) = 5.0$ and $F'(2.0) = 0.6$. Explain how this problem uses a linear approximation.

P5.18 An investment account is set up with an initial deposit of \$1000, and it grows at an annual rate of 5%.

(a) Write an equation for the amount A in the account as a function of time t in years.

(b) Find the *doubling time* for this account, the time it takes for the amount in the account to double.

(c) If you solved part (b) with algebra, you had to solve the equation $2 = (1.05)^t$. Explain. Since you are solving for an exponent, you would need to use logarithms since a logarithm is the inverse of an exponential; thus, the logarithm of an exponential expression yields an exponent. For this particular equation, $\ln 2 = t \ln 1.05$ (Why?) Complete the work of finding the doubling time t from this equation.

(d) In part (c) we happened to use a logarithm with base e, but we could have used any base logarithm. We often use base e or base 10 logarithms because those functions are readily available on calculators. What if we had used a base 1.05 logarithm applied to both sides of the equation $2 = (1.05)^t$? What equation would we get for t?

(e) The solutions in parts (c) and (d) suggest the equality $\dfrac{\ln 2}{\ln 1.05} = \log_{1.05} 2$. Verify that this is true.

P5.19 (Continuation) The equation $a = b^x$ can be solved for x using the two approaches shown in the previous problem.

(a) Write out two possible solutions, one using a logarithm with a base c, the other using a logarithm with a base b.

(b) Equate the two expressions for x in part (a) to yield the *change of base* formula:

$$\log_b a = \frac{\log_c a}{\log_c b}.$$

Prelaboratory 5: Transformations and Derivatives

PL5.1 Graph the following functions on the same set of axes.

- **(a)** $y = x^2$
- **(b)** $y = x^2 + 1$
- **(c)** $y = x^2 + 4$
- **(d)** $y = x^2 - 3$
- **(e)** How are these 4 graphs related?
- **(f)** What is the effect of the transformation $y = f(x) + k$ (where k is a constant number) on the graph of $y = f(x)$?

PL5.2 Graph the following functions on the same set of axes.

- **(a)** $y = \sin x$
- **(b)** $y = 3 \sin x$
- **(c)** $y = -\sin x$
- **(d)** $y = \frac{1}{2} \sin x$
- **(e)** How are these 4 graphs related?
- **(f)** What is the effect of the transformation $y = af(x)$ (where a is a constant number) on the graph of $y = f(x)$?

PL5.3 Graph the following functions on the same set of axes.

- **(a)** $y = \sqrt{x}$
- **(b)** $y = \sqrt{x - 4}$
- **(c)** $y = \sqrt{x + 1}$
- **(d)** How are these 3 graphs related?
- **(e)** What is the effect of the transformation $y = f(x - h)$ (where h is a constant number) on the graph of $y = f(x)$?

PL5.4 Graph the following functions on the same set of axes.

- **(a)** $y = \cos(x)$
- **(b)** $y = \cos(2x)$
- **(c)** $y = \cos(\frac{1}{3}x)$
- **(d)** How are these 3 graphs related?
- **(e)** What is the effect of the transformation $y = f(mx)$ (where m is a constant number) on the graph of $y = f(x)$?

Laboratory 5: Transformations and Derivatives

L5.1 *Vertical shift.* Graph the function $f(x) = x^2$ and the tangent line to $f(x)$ at $x = 1$. Now refer to the graphs from #PL5.1 of the three functions $f(x) = x^2 + k$ for $k = 1, 4,$and -3. Notice that the new functions are vertical shifts of the original function. Consider how a vertical shift by k units affects the equation of the tangent line at $x = 1$. Write equations for the tangent lines at $x = 1$ for the 3 functions that are vertical shifts of the original function. Check your conclusions by adding your equations for the tangent lines to your graphs in a graphing calculator.

L5.2 (Continuation) What do all of the tangent lines have in common? How does your answer relate to the derivative of $f(x) = x^2 + k$ at $x = 1$?

(a) In general, how do the derivatives of a function and a vertical shift of that function, obtained by adding a constant to the function, relate to each other?

(b) Find the derivatives of the following functions.

i. $y = x^2 + 5$

ii. $y = -2 + \cos x$

iii. $y = f(x) + k$

L5.3 *Vertical stretch/shrink.* Graph the function $f(x) = \sin x$ and the tangent line to $f(x)$ at $x = \frac{\pi}{4}$. Now refer to the graphs from #PL5.2 of the three functions $f(x) = a \sin x$ for $a = 3, -1,$ and $\frac{1}{2}$. Notice that the new functions are vertical stretches or shrinks of the original function. Consider how a vertical stretch/shrink by a units affects the equation of the tangent line at $x = \frac{\pi}{4}$. Write equations for the tangent lines at $x = \frac{\pi}{4}$ for the three functions that are vertical stretches/shrinks of the original function. Check your conclusions by adding your equations for the tangent lines to your graphs in a graphing calculator.

L5.4 (Continuation) How are all of the slopes of the tangent lines related? How does your answer relate to the derivative of $f(x) = a \sin x$ at $x = \pi/4$?

(a) In general, how do the derivatives of a function and a vertical stretch or shrink of that function, obtained by multiplying the function by a constant, relate to each other?

(b) Find the derivatives of the following functions.

i. $y = 5x^2$

ii. $y = -2 \cos x$

iii. $g(x) = a \cdot f(x)$

L5.5 *Horizontal shift.* Graph the function $f(x) = \sqrt{x}$ and the tangent line to $f(x)$ at $x = 4$. Now refer to the graphs from #PL5.3 of the two functions $f(x) = \sqrt{x-h}$ for $h = 4, -1$. Notice that the new functions are horizontal shifts of the original function, and so the tangent line also has a horizontal shift. Consider how a horizontal shift by h units affects the equation of the tangent line that was originally tangent at $x = 4$. Write equations for the tangent lines that are shifted horizontally from $x = 4$ when you graph the two functions that are horizontal shifts by 4 and -1 of the original function. Check your conclusions by adding your equations for the tangent lines to your graphs in a graphing calculator.

L5.6 (Continuation) What do all of the tangent lines have in common? How does your answer relate to the derivative of $f(x) = \sqrt{x-h}$ at $x = 4$?

(a) In general, how do the derivatives of a function and a horizontal shift of that function, obtained by subtracting a constant from the x value, relate to each other?

(b) Find the derivatives of the following functions.

i. $y = (x+5)^2$

ii. $y = \cos\left(x - \frac{\pi}{4}\right)$

iii. $g(x) = f(x-h)$

L5.7 *Horizontal stretch/shrink.* Graph the function $f(x) = \cos x$ and the tangent line to $f(x)$ at $x = \frac{\pi}{2}$. Now refer to the graphs from #PL5.4 for the function $f(x) = \cos(mx)$ when $m = 2$ and $m = \frac{1}{3}$. Notice that the new functions are horizontal stretches or shrinks of the original function, and the point of tangency also moves from its original location at $x = \frac{\pi}{2}$. For example, when graphing $f(x) = \cos(2x)$, notice that the point of tangency moves from $x = \frac{\pi}{2}$ to $x = \frac{\pi}{4}$. Why do you think that is?

Consider how a horizontal stretch/shrink by a factor of m affects the equation of the tangent line that was originally tangent at $x = \frac{\pi}{2}$. Write equations for these transformed tangent lines for the two functions that are horizontal stretches/shrinks of the original function. Check your conclusions by adding your equations for the tangent lines to your graph in a graphing calculator.

L5.8 (Continuation) How are the slopes of the tangent lines related? How does your answer compare to the derivative of $f(x) = \cos x$ at $x = \frac{\pi}{2}$?

(a) In general, how do the derivatives of a function and a horizontal stretch of that function, obtained by multiplying x by a constant, relate to each other?

(b) Find the derivatives of the following functions.

i. $y = \sqrt{4x}$

ii. $y = \sin(2x)$

iii. $g(x) = f(mx)$

L5.9 Construct a table that summarizes the relationships between the derivative of a function and the derivatives of the transformations of the function investigated in this lab. Write a sentence explaining each row of the table: When a function is transformed by __________, the derivative is affected by __________ because __________.

P6.1 Find the values of the expressions $\lim_{h\to 0} \frac{e^h - 1}{h}$ and $\lim_{k\to 1} \frac{\ln k}{k-1}$. Show that each value can be interpreted as a slope, and thus as a derivative.

P6.2 Each of the following represents a derivative. Use this information to evaluate each limit.

(a) $\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$ **(b)** $\lim_{h\to 0} \frac{1}{h}\left(\sin\left(\frac{\pi}{6}+h\right) - \sin\left(\frac{\pi}{6}\right)\right)$ **(c)** $\lim_{x\to a} \frac{e^x - e^a}{x-a}$

P6.3 In previous labs and problem sets, we learned that the derivative of $y = \sqrt{x}$ is $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. In this problem we will see how to obtain this derivative from the limit definition of the derivative.

(a) Explain why $\sqrt{x}$ has the following limit expression as its derivative.

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

(b) Re-express the limit formula in part (a) by using the technique of multiplying the numerator and denominator by the same expression $\sqrt{x+h} + \sqrt{x}$. Simplify and cancel common factors.

(c) Taking the limit at h approaches 0 of the expression in part (b), obtain the expression for the derivative of $\sqrt{x}$.

P6.4 Find the approximate values for both $g(2.1)$ and $g(1.85)$, given that $g(2.0) = -3.5$ and

(a) $g'(2.0) = 10.0$

(b) $g'(2.0) = -4.2$

P6.5 Find the derivatives of the following functions that have undergone a combination of transformations. Your answers should look like "$f'(x) = ...$", or "$y' = ...$", or a similar form.

(a) $y = e^{2x-2}$

(b) $p = 3960 \cdot e^{0.15t} + 5280$

(c) $g(x) = \frac{5}{x+2}$

(d) $y = 1 + 3\sqrt{x-4}$

(e) $y = -1 + \cos(3x + \pi)$

(f) $h(x) = (5x+1)^3$

P6.6 After being dropped from the top of a tall building, the height of an object is described by $y = 400 - 16t^2$, where y is measured in feet and t is measured in seconds. Find a formula for the rate of descent (in feet per second) for this object. Your answer will depend on t. How fast is the object falling after 2 seconds?

P6.7 Calculate derivatives for $A(r) = \pi r^2$ and $V(r) = \frac{4}{3}\pi r^3$. The resulting functions A' and V' should look familiar. Could you have anticipated their appearance? Explain.

P6.8 (Continuation) If a is a number, then $f'(a)$ is the derivative of $f(x)$ evaluated at a. This means that to find $f'(a)$, you first determine a formula for $f'(x)$ and then substitute the value of a for x.

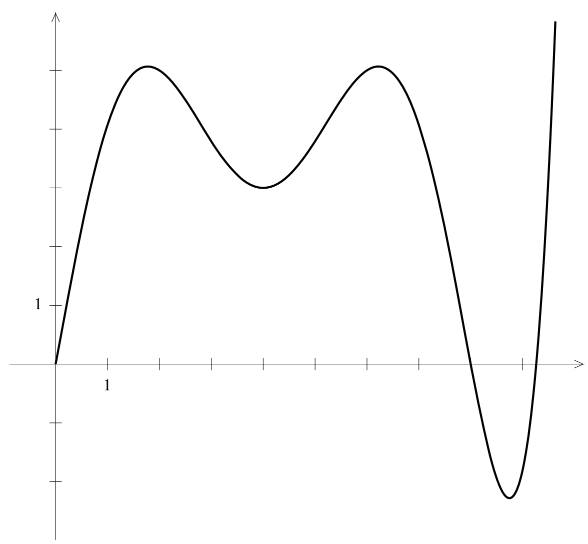
(a) Find the values of $A'(2)$ and $V'(2)$.

(b) Explain the geometric meaning of the quantities found in part (a).

P6.9 Interpret the diagram as a *velocity-time* graph for an object that is moving along a number line. The horizontal axis represents time (in seconds) and the vertical axis represents velocity (in meters per second).

(a) The point (8.8,-2.2) is on the graph. Find it in the diagram and describe what is going on around that point. In particular, what is the significance of the sign? Choose three other conspicuous points on the graph and interpret them.

(b) Suppose the object starts its journey when $t = 0$ at a definite point P on the number line. Use the graph to estimate the position of the object (in relation to P) two seconds later.



P6.10 (Continuation) On a separate set of axes, sketch the derivative of the function whose graph appears above. Interpret your graph in this context.

P6.11 You have found the derivatives of certain power functions. The derivative of $f(x) = x^2$ is $f'(x) = 2x$, and the derivative of $g(x) = x^3$ is $g'(x) = 3x^2$.

(a) What do you think is a formula for the derivative of a power function $y = x^n$?

(b) Look at the graphs of some other power functions (such as $x^4, x^{-2}, \sqrt{x}$, and so on) and their derivatives. [Desmos is an app that allows you to graph the derivative when the operator $\dfrac{d}{dx}$ is applied to a function of x.] See if your guess from part (a) is correct, and if not, modify your formula.

P6.12 The *linear approximation* for $(1+x)^k$:

(a) Find the linear approximation for $(1+x)^2$ centered at $a=0$. Verify graphically that you have indeed found the tangent line approximation for this function.

(b) Find the linear approximation for $(1+x)^3$ centered at $a=0$. Verify graphically.

(c) Find the linear approximation for $\sqrt{1+x}$ centered at $a=0$. Verify graphically.

(d) Justify the linear approximation formula $(1+x)^k \approx 1+kx$.

P6.13 Illuminated by the parallel rays of the setting Sun, Andy rides alone on a merry-go-round, casting a shadow that moves back and forth on a wall. The merry-go-round takes 9 seconds to make one complete revolution, Andy is 24 feet from its center, and the Sun's rays are perpendicular to the wall. Let N be the point on the wall that is closest to the merry-go-round.

(a) Interpreted in radian mode, $f(t) = 24\sin\left(\dfrac{2\pi}{9}t\right)$ describes the position of the shadow relative to N. Explain.

(b) Calculate the speed (in feet per second) of Andy's shadow when it passes N, and the speed of the shadow when it is 12 feet from N.

P6.14 Find the derivative of $\ln(3x)$. Explain why this function can be viewed as a vertical shift or a horizontal shrink of the function $\ln x$. Show that both approaches lead to the same derivative.

P6.15 Find the derivative of e^{x+5}. Explain why this function can be viewed as a horizontal shift or a vertical stretch of the function e^x. Show that both approaches lead to the same derivative.

P6.16 Recall the change-of-base formula for logarithms: $\log_b a = \dfrac{\log_c a}{\log_c b}$. Apply this formula to rewrite the following logarithms in terms of natural logarithms.
(a) $\log_2 x$, **(b)** $\log_{10}(x+3)$, **(c)** $\log_3(x^2)$.

P6.17 (Continuation) Use the strategy in the previous number to find the derivative formula for any base logarithm function by answering parts (a)-(c) below.

(a) What is the derivative of $\ln x$? What is the derivative of $\dfrac{\ln x}{\ln 3}$? What is the derivative of $\log_3 x$?

(b) Rewrite $\log_b x$ in terms of $\ln x$ by using the change-of-base formula.

(c) Find the derivative of $\log_b x$ by using the expression found in part (b).

(d) Find the derivatives of the expressions given in the previous problem: $\log_2 x$, $\log_{10}(x+3)$, and $\log_3(x^2)$.

P6.18 By now you know that the derivative of $y = e^x$ is $y' = e^x$. To find the derivative of other exponential functions, such as 2^x, it is helpful to rewrite the exponential as a base e exponential. This can be accomplished by recognizing that 2 is equal to some power of e or in symbols, $2 = e^k$.

(a) Find the exact solution for k in the equation $2 = e^k$ using natural logarithms.

(b) Using your previous answer, show how to rewrite 2^x as e^{kx}.

(c) Find the derivative of 2^x by finding the derivative of the equivalent expression e^{kx}.

(d) Write a general formula for the derivative of b^x, for any base $b > 0$.

P6.19 (Continuation) Find the derivatives of the following functions.
(a) $y = 2^{x+5}$, **(b)** $A = 1000 \cdot (1.05)^t$, **(c)** $T = 68 + 132 \cdot (0.9)^t$.

P6.20 *Simple harmonic motion.* An object is suspended from a spring, 40 cm above a laboratory table. At time $t = 0$ seconds, the object is pulled 24 cm below its equilibrium position and released. The object bobs up and down thereafter. Its height y above the laboratory table is described, with t measured in radians, by $y = 40 - 24\cos(2\pi t)$.

(a) What is the period of the resulting motion?

(b) Find the average velocity of the object during the first 0.50 second of motion.

(c) Find the instantaneous velocity of the object when $t = 0.25$ second. Find a way of convincing yourself that the object never moves any faster than it does at this instant.

Prelaboratory 6: Addition Rule and Product Rule for Derivatives

PL6.1 For a function $f(x)$, write a formula for the linear approximation centered at $x = a$.

PL6.2 Use technology to graph the lines $f(x) = -(x-3)+2$ and $g(x) = \frac{1}{2}(x-3)+2$. You should notice from the formulas and from the graph that these two lines intersect at $(3, 2)$. Now graph the function that is the sum of the two lines $f(x) + g(x)$. Confirm that the sum function is also a line. What is its slope? How does the slope of the sum of two lines relate to the slope of the individual lines?

PL6.3 For the same lines $f(x) = -(x - 3) + 2$ and $g(x) = \frac{1}{2}(x - 3) + 2$, graph the function that is the product of the two lines $f(x)g(x)$. This product is not a line (what is it?), but you can estimate the slope at the point with $x = 3$. What is its slope? Is the slope of the product of two lines equal to the product of the slopes of the individual lines?

Laboratory 6, Part 1: Addition Rule

L6.1 For the function $f(x) = x^2 + 1$, what is the linear approximation centered at $x = 1$? Use technology to graph $f(x)$ and its linear approximation. For the function $g(x) = \frac{1}{x}$, what is the linear approximation centered at $x = 1$? Use technology to show $g(x)$ and its linear approximation on the same axes as f.

L6.2 Now graph the sum of the functions $f(x) + g(x)$ and the line that is the sum of the two linear approximations centered at $x = 1$. Does the sum of the linear approximations for f and g appear to be the linear approximation for the function $f + g$?

L6.3 We know from our previous work that the general equations for the linear approximations for two functions f and g centered at $x = a$ are $f(x) \approx f'(a)(x-a)+f(a)$ and $g(x) \approx g'(a)(x - a) + g(a)$.

- **(a)** Combine these two lines to find a linear approximation for the sum function $f(x) + g(x)$ centered at the point with $x = a$. What is the slope of this linear approximation?
- **(b)** How does this slope relate to the derivatives of $f(x)$ and $g(x)$ at $x = a$?
- **(c)** Write a statement for the Addition Rule for derivatives relating the derivative of the sum of two functions to the derivatives of each function.

L6.4 Apply the Addition Rule to find the derivatives of the following functions.

- **(a)** $y = x^2 + \frac{1}{x}$
- **(b)** $y = x + \sin(x)$

Laboratory 6, Part 2: Product Rule

L6.5 Graph the function $f(x) = (x-2)^2 + 1$ and its linear approximation $l(x)$ centered at the point $(3, 2)$. On the same set of axes, graph $g(x) = \dfrac{2}{x-1} + 1$ and its linear approximation $m(x)$ centered at $(3, 2)$. Use an appropriate window in the first quadrant.

L6.6 Now graph the products $p(x) = f(x)g(x)$ and $q(x) = l(x)m(x)$. What do you notice about the slopes of the curves $p(x)$ and $q(x)$ around $x = 3$?

L6.7 What is the slope of the product $p(x)$ at the point with $x = 3$? Use your graph to find the slope of the tangent to the curve at this point. This slope is the derivative of the product function at $x = 3$.

L6.8 What is the slope of the linear approximation $l(x)$ at $(3, 2)$? This is the derivative of $f(x)$ at $x = 3$. What is the slope of the linear approximation $m(x)$ at $(3, 2)$? This is the derivative of $g(x)$ at $x = 3$. Does the derivative of the product function at $x = 3$ equal the product of the derivatives of $f(x)$ and $g(x)$?

L6.9 The answers to the previous problem shows that the derivative of a product does not equal the product of the derivatives. There is something more complicated going on here, to which the following steps lead.

(a) Write the formula for the linear approximation of a function f centered at the point with $x = a$, and write the formula for the linear approximation of a function g centered at the point with $x = a$.

(b) Multiply the linear approximations of f and g from part (a). Notice that this is a quadratic function.

(c) Verify that the derivative of the quadratic from part (b) is the linear expression $2f'(a)g'(a)(x-a) + f(a)g'(a) + g(a)f'(a)$.

(d) What is the value of this expression at the point with $x = a$? Explain why this is also the value of $p'(a)$, which is $f(a)g'(a) + g(a)f'(a)$.

(e) The previous expression leads to the Product Rule for the derivative of $p(x) = f(x)g(x)$: $p'(x) = f(x)g'(x) + g(x)f'(x)$ State the product rule in words: The derivative of a product of 2 functions is the first __________ the derivative of the __________ plus __________ times the derivative of __________.

L6.10 Use the Product Rule to find the derivatives of the following functions.

(a) $y = x\sin(x)$

(b) $y = x^2 e^x$

L6.11 Write a one-page summary of your results from this lab: statements of the Addition Rule and the Product Rule, in symbols and in words; results of the rules applied to the functions given in L4 and L9; and, application of the rules to two other functions that illustrate how your toolkit of derivatives has expanded.

P7.1 If $y = \sin x$, then the derivative equation $y' = \cos x$ can also be expressed with the notation $\dfrac{dy}{dx} = \cos x$. Show where this notation comes from by writing a limit that relates $\dfrac{dy}{dx}$ and $\dfrac{\Delta y}{\Delta x}$. Is $\dfrac{dy}{dx}$ a ratio of two numbers in the usual sense? The form $\dfrac{dy}{dx}$ for the derivative is known as *Leibniz notation.*

P7.2 *Oscillations about a line.*

(a) Graph the functions $f(x) = x + \sin x$ and $g(x) = x$.

(b) Find the derivative of f.

(c) What is the slope of the graph of f at the points where $f(x) = g(x)$?

(d) Which of the following characteristics apply to the graph of f? (1) periodic; (2) always increasing; (3) alternately increasing and decreasing; (4) non-decreasing; (5) rotated sinusoidal; (6) horizontal tangent at x values equal to odd multiples of π

P7.3 Find the derivative of each of the following functions.

(a) $f(x) = x + \ln x$

(b) $g(t) = 3t - 5\sin t$

(c) $L(x) = \sqrt{4 - x}$

(d) $P(t) = 12 + 4\cos(\pi t)$

P7.4 Find a function that fits the description $f'(t) = -0.42f(t)$. There are many from which to choose.

P7.5 Assuming A and C are constants, find the derivatives of the following functions: **(a)** $y = A \cdot e^x$; **(b)** $y = e^x + C$. For which of these functions is it true that $\dfrac{dy}{dx} = y$?

P7.6 Given that f is a differentiable function and that the value of c does not depend on x, explain the following differentiation properties:

(a) If $g(x) = f(x - c)$, then $g'(x) = f'(x - c)$.

(b) If $g(x) = c \cdot f(x)$, then $g'(x) = c \cdot f'(x)$.

(c) If $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$.

P7.7 The PEA Ski Club is planning a ski trip over a long weekend. They have 40 skiers signed up to go, and the ski resort is charging \$180 per person. The resort manager offers to reduce the group rate of \$180 per person by \$3 for each additional registrant as long as revenue continues to increase.

(a) Calculate how much money (revenue) the resort will receive if no extra students sign up beyond the original 40. How much is the revenue if 5 extra students sign up?

(b) Let x be the number of additional registrations beyond the original 40. In terms of x, write expressions for $p(x)$, the total number of people going, and for $q(x)$, the cost per person. Revenue is the product of the number of people and the cost per person, so the revenue function is $r(x) = p(x) \cdot q(x)$.

(c) Use the product rule to find the derivative $r'(x)$ in terms of $p(x)$, $p'(x)$, $q(x)$, and $q'(x)$.

(d) Find the derivative $r'(x)$ by first multiplying out the product of $p(x)$ and $q(x)$, then using the Addition Rule. Compare your answer with the answer from part (c). What value of x yields the maximum revenue.

P7.8 Find the derivative of each of the following functions.

(a) $f(x) = x^2 + x^{-2}$

(b) $g(t) = \sqrt{5t}$

(c) $y = (1 + x)^n$

P7.9 A roller coaster descends from a height of 100 ft on a track that is in the shape of the parabola $y = 100 - \frac{1}{4}x^2$, where y is the height in feet and x is the horizontal distance in feet from the point of maximum height.

(a) What is the slope of the track at any point (x, y)? Where will the track run into the ground and at what angle?

(b) To allow the ride to transition smoothly to the ground, another piece of track in the shape of the parabola $y = a\,(x - 40)^2$ is joined to the previous track and a horizontal track at $y = 0$. What is the slope of this track assuming a is a constant?

(c) Using a graphing app, find an approximate value for a that fits the tracks together to give a smooth ride from the top to the ground.

(d) Use your knowledge of calculus to find the exact value of a that smoothly fits the tracks together.

P7.10 An object moves along the x-axis according to the equation $x(t) = 4t - t^2$.

(a) Obtain a graph of $x(t)$ versus t. Explain what the height of any point above the horizontal axis on the graph represents.

(b) Employ *differentiation* (which is the name of the process for finding a derivative) to find a formula for the velocity of the object.

(c) Use this derivative to find the velocity and speed of the object each time it passes the point $x = 0$.

P7.11 Find the derivative of each of the following functions.

(a) $y = \sin x \cos x$

(b) $A(x) = x \cdot e^x$

(c) $P(u) = (u + 2)^5$

P7.12 Consider the equations $y'(t) = 0.12$ and $p'(t) = 0.12 \cdot p(t)$. They say similar but different things about the functions whose rates of change they are describing.

(a) For each equation, find a function with the given derivative. There are many possible answers for each one.

(b) For each equation, find the particular function that has a value of 36 when $t = 0$.

P7.13 The population of Halania is increasing at a rate of 1.3% per year while per capita energy consumption is increasing at a constant rate of 8×10^6 BTUs per year.

(a) Explain why the functions $P(t) = 23(1.013)^t$ and $E(t) = 8t + 150$ are reasonable models for population and per capita energy consumption, where P is in millions of people, E is in millions of BTUs, and t is in years since 2010. What are the meanings of the constants 23, 1.013, 8, and 150 in these models?

(b) Write an expression for total energy consumption $T(t)$, which is the product of population and per capita energy consumption. Use the product rule to find the derivative of this function.

(c) How fast will total energy consumption be changing at the beginning of 2020? What is the percent change at that time?

(d) Obtain a graph of the percent change in total energy consumption. Comment on the shape of this graph.

P7.14 A particle moves along a line according to $p = t^4 - 4t^3 + 3$, during the time interval $-1 \le t \le 4$. Calculate the velocity function $\dfrac{dp}{dt}$ and the acceleration function $\dfrac{d^2p}{dt^2}$. Use them to help you give a detailed description of the position of the particle according to the following questions.

(a) At what times is the particle (instantaneously) at rest, and where does this happen?

(b) During what time intervals is the position p increasing? When is p decreasing?

(c) At what times is the acceleration of the particle zero? What does this signify?

(d) What is the complete range of positions of the particle?

(e) What is the complete range of velocities of the particle?

Prelaboratory 7: The Most Exciting Moment on the Tilt-a-Whirl

PL7.1 An object travels counterclockwise at 1 unit per second around the unit circle starting at the point $(1, 0)$.

(a) Write parametric equations for the position of the object in terms of t, the time in seconds. Use these equations to represent the position of the object as a vector with x- and y-components given as functions of time t in seconds. You can think of the position of the object as the tip of a vector with its tail at the origin.

(b) Find the *velocity vector* at the instant when the object is at each of the following points: $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. How is the velocity vector oriented relative to the position vector?

PL7.2 (Continuation) Find the velocity vector in terms of t for the object by taking derivatives of the x- and y-components found in part (a). Obtain a sketch of the position and velocity vectors at the following points: $(1, 0)$, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and $(-0.6, 0.8)$. Confirm that the velocity vector has a constant orientation relative to the position vector.

PL7.3 (Continuation) Suppose the object is actually tethered to a string of unit length as it is spun around a circle. What would happen to the object if the string suddenly broke when the object reached $(-0.6, 0.8)$?

PL7.4 In most cases, a derivative, f', has its own derivative, f'', which is useful in many contexts. This function is called the *second derivative of f.* If $y = f(x)$, then we can also write the second derivative as $\frac{d^2y}{dx^2}$. For each of the following, calculate the first and second derivatives.

(a) $y = \sin x$ **(b)** $s(t) = 400 - 16t^2$

PL7.5 Find the acceleration vector for the circular function expressed with parametric equations in PL7.1. This requires the derivatives of the components of the velocity vector.

PL7.6 **(a)** Confirm that the parametric equations $x = 3\cos t$, $y = 5\sin t$ represent an ellipse.

(b) Write the position vector, velocity vector, and acceleration vector.

(c) The magnitude of the velocity vector is the speed of the object, which can be found by using the Pythagorean theorem: speed $= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$. Find the speed as a function of time for an object moving on this ellipse.

Laboratory 7: The Most Exciting Moment on the Tilt-a-Whirl

The Tilt-a-Whirl is a popular carnival ride in which riders sit in carts that can be spun by the rider in circles, while the carts themselves are going around in a larger circle. The distance from the center of the large circle to the center of a cart's small circle is 5 meters. The cart's small circle has a one meter radius from center to seat. It takes 12 seconds to go once around the large circle. The small circles are actually controlled by the riders, but suppose that a rider is spinning at a constant rate of once every three seconds.

We can describe the location (x, y) of a rider by setting up our coordinate system with the center of the large circle at the origin. We want to know the location of a rider as a function of time, and we will use parametric equations where x and y are functions of time t. Recall that in Math 3-4 we used parametric equations to describe the location of a point on a rotating wheel (a Ferris wheel, for example), and we modeled the location of a point on a wheel that rolls along the ground. Both situations are helpful references for modeling the Tilt-a-Whirl, which is like a rotating wheel — at least the large circle is — but with the addition of a small circle that spins as it rotates around the large circle.

Part 1

L7.1 The Tilt-a-Whirl starts up and you are in one of the carts. Sketch a graph of what you think your path looks like as the large circle turns around its center (which is the origin of your coordinate system), and your cart is simultaneously spinning around its small circle.

L7.2 Write parametric equations for $Q(x, y)$, where Q is the center of a cart's small circle as the large circle turns around the origin. Use the relevant values given in the first paragraph.

L7.3 Write parametric equations for the motion of $R(x, y)$, where R is a point on a cart's small circle, and the motion is expressed relative to the center of the small circle. Equivalently, find the equations for R assuming the large circle is not moving. Use the relevant values given in the first paragraph of the introduction.

L7.4 Explain why the parametric equations for $P(x, y)$, where P represents the location of a cart as its spins around the small circle while also rotating around the large circle, can be found by adding Q and R, as in $P(x, y) = Q(x, y) + R(x, y)$.

L7.5 Obtain a graph of $P(x, y)$ using graphing technology. How does your graph compare with the graph you sketched in number 1? Adjust your equations as necessary to get a reasonable result.

Part 2

L7.6 You can create a position vector $\mathbf{s}$ from the origin to the point P, the magnitude of which gives you the rider's distance from the origin. The components of $\mathbf{s}$ are the parametric functions $x(t)$ and $y(t)$ that you found in Part 1 when you graphed $P(x, y)$. Write an equation in terms of $x(t)$ and $y(t)$ for the magnitude of $\mathbf{s}$, and graph the magnitude versus time for one 12-second revolution, $0 \leq t \leq 12$.

L7.7 The velocity can also be represented as a vector, which we will call $\mathbf{v}$. The velocity has a magnitude and a direction, and the components of $\mathbf{v}$ are the derivatives of the components of $\mathbf{s}$. Use calculus to find the velocity vector $\mathbf{v} = [x'(t), y'(t)]$.

L7.8 The magnitude of the velocity vector is the speed of the rider, which is given by $|\mathbf{v}| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$. Graph the speed over the 12-second interval of a single revolution.

L7.9 Now do the same thing for acceleration: find the acceleration vector using derivatives and graph its magnitude as a function of time.

L7.10 You now have three graphs that you can align as functions of time, or you can plot all three on the same set of axes. Based on these graphs, as well as the graph you generated in Part 1, at what points on the Tilt-a-Whirl do you think riders have the most fun? Explain your reasoning.

Part 3

L7.11 Open the accompanying Geogebra file and run the animations in the graphing windows. Compare the graph in the window at the bottom of the screen, which shows the magnitudes of the position/velocity/acceleration vectors, to the graphs you found in Part 2.

L7.12 In the top window, you will notice a point moving around the Tilt-a-Whirl curve, as well as vectors for position, velocity, and acceleration. The position vector has its tail at the origin, the tail of the velocity vector is anchored to the tip of the position vector, and likewise the acceleration vector is attached to the velocity vector. Study the animation in conjunction with the graph of the magnitudes of the vectors. Now reconsider an earlier question: At what points on the Tilt-a-Whirl do you think riders have the most fun?

L7.13 Write a report including the relevant equations (and how you derived them), graphs, diagrams of the position-velocity-acceleration vectors, and an explanation of how parametric equations and vectors and their derivatives are related. Be sure to discuss your reasoning about the Most Exciting Moment on the Tilt-a-Whirl.

Extension

As an additional challenge, investigate what happens if you speed up the rotation of the cart. One way you can do this in the Geogebra simulation is by introducing a slider parameter to the vector equations in place of the number 3 that corresponds to the spinning rate of the cart. If k is the parameter, then the cart spins once every k seconds instead of once every 3 seconds.

P8.1 What can be said about the derivative of
(a) an *odd* function? **(b)** an *even* function?

P8.2 Graph the function $y = |x^2 - 4|$ over the interval $-4 \le x \le 4$.

(a) What is the equation of the tangent line at the point with $x = 0$? If your definition of tangent line is based upon the phrase "only one point of intersection," then is there another line that intersects the curve only once at the point with $x = 0$?

(b) Come up with a definition of tangent line that results in a unique line for x-values on the curve such that $-4 \le x \le 4$. You will want to consider how "local linearity" can be used in your definition.

(c) Notice that this graph is not "smooth" at the points where $x = \pm 2$. Can you find a unique tangent line to the curve at either of those points? Based on your definition in part (b), can you say that a tangent line exists at either of those points? How does the existence of a tangent line at a point rate to *differentiability*, the existence of the derivative at that point?

P8.3 Suppose the revenue and cost functions (in thousands of dollars) for a manufacturer are $r(x) = 10x$ and $c(x) = x^3 - 3x^2 + x$, where x is in thousands of units.

(a) What amount should be produced to generate the most profit? [Note: Profit equals revenue minus cost.]

(b) Compare the slopes of the revenue and cost functions at the x-value found in (a).

P8.4 (Continuation) Graph the revenue and cost functions from the previous problem. The location of the maximum profit, which is the greatest difference between $r(x)$ and $c(x)$, appears to occur where the slopes of the two functions are equal. Use the derivative of profit $p(x) = r(x) - c(x)$ to show that this is true in general.

P8.5 Use what you know about derivatives of power functions (functions of the form $y = x^n$) to find the derivative of $Q(x) = 60 - \dfrac{12}{x\sqrt{x}}$

P8.6 Find the first and second derivative of $g(x) = \dfrac{e^x}{x}$.

P8.7 Obtain a graph of $y = \ln(1 - x)$. What is the linear approximation centered at $x = 0$?

P8.8 Calculate the derivative of each of the following functions:
(a) $f(t) = t^2 e^{-t}$, **(b)** $g(u) = u^2\sqrt{u}$, **(c)** $y = x \ln x$.

P8.9 The linear equation $y - 5 = m(x - 2)$ represents a family of lines.

(a) Describe the family of lines.

(b) What values of m will form a right triangle in the first quadrant bounded by the axes and the line?

(c) Find expressions for the x- and y-intercepts in terms of m.

(d) Solve for the value of m that minimizes the area of the triangle in (b).

P8.10 Just as the second derivative is the derivative of the first derivative, the third derivative is the derivative of the second derivative, and so on. For the following functions, find the first and second derivatives, and continue finding higher derivatives until you are able to deduce a formula for the nth derivative.

(a) $y = t \cdot e^t$, **(b)** $f(x) = x^n$.

P8.11 An object moves along the x-axis with its position at time t given by $x = t + 2\sin t$, for $0 \leq t \leq 2\pi$.

(a) What is the velocity $\dfrac{dx}{dt}$ of the object at time t?

(b) For what values of t is the object moving in the positive direction?

(c) For what values of t is the object moving in the negative direction?

(d) When and where does the object reverse its direction?

P8.12 For each of the following description of $\dfrac{dy}{dx}$, with constants c and k, what type of function is y? For example, your answer in (a) could be something like this: *The derivative is a constant, therefore the function must be* ____________.

(a) $\dfrac{dy}{dx} = c$ **(b)** $\dfrac{dy}{dx} = ky$ **(c)** $\dfrac{dy}{dx} = \cos x$

P8.13 (Continuation) Equations involving derivatives, of which these are examples, are known as *differential equations*. Give a particular solution of y as a function of x for each one. Why are there many solutions for each equation?

Mathematics 42C

Mathematics Department
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Prelaboratory 8: Graphs and the Derivative

PL8.1 Find the first derivative $\frac{dy}{dx}$ and the second derivative $\frac{d^2y}{dx^2}$ for each of the following functions.

(a) $y = x^3 - x$

(b) $y = x^4 - x^2$

(c) $y = \cos x$

(d) $y = x^2 - \frac{1}{x}$

PL8.2 A quadratic function F is defined by $F(x) = ax^2+bx+c$, where a, b, and c are constants and a is nonzero. Find the derivative of F, and then find the value of x that makes $F'(x) = 0$. The corresponding point on the graph $y = F(x)$ is special. Why?

PL8.3 For the function $f(x) = x^2 - x^3$,

(a) Use a graph to confirm that f has locally extreme points at (0,0) and approximately (0.667, 0.148). What is the slope of the curve at these two points?

(b) Use a derivative to show that the locally extreme points are exactly (0,0) and $(\frac{2}{3}, \frac{4}{27})$.

Lab 8: Graphs and the Derivative

L8.1 Let $y = x^3 - x$. Find the derivative $\frac{dy}{dx}$. Graph both functions on the same set of axes.

(a) What do you notice about the graph of y for those x-values where $\frac{dy}{dx} = 0$?

(b) What do you notice about the graph of y for those x-values where $\frac{dy}{dx} > 0$?

(c) What do you notice about the graph of y for those x-values where $\frac{dy}{dx} < 0$?

L8.2 Repeat #L8.1 for the following functions.

(a) $y = \sin x$

(b) $y = x^4 + 2x^3 - 7x^2 - 8x + 12$

(c) $y = x - \frac{1}{x}$

L8.3 Fill in the blanks in the following statements based on your findings.

(a) If the first derivative is positive, then the function is ________________.

(b) If the first derivative is negative, then the function is ________________.

(c) If the first derivative changes sign (goes from positive to negative or negative to positive), then the function has a ________________.

L8.4 An 8" by 15" rectangular sheet of metal can be transformed into a rectangular box by cutting four congruent squares from the corners and folding up the sides. The volume $V(x)$ of such a box depends on x, the length of the sides of the square cutouts.

- **(a)** Use algebra to find an expression for the volume $V(x)$. For what values of x does $V(x)$ make sense?
- **(b)** Use $V'(x)$to find the largest value of $V(x)$ and the x-value that produces it. Explain your reasoning. Use a graph of $V(x)$ to confirm your answer.

L8.5 Let $y = x^4 - x^2$. Find the derivative $\dfrac{dy}{dx}$ and the second derivative $\dfrac{d^2y}{dx^2}$. Graph all three functions on the same set of axes in a suitable sized window centered at the origin.

- **(a)** What do the values of $\dfrac{d^2y}{dx^2}$ tell you about the graph of $\dfrac{dy}{dx}$? This is an application of the concepts from the first part of this lab. You should include in your explanation words such as increasing, decreasing, turning points, and so on.
- **(b)** What do the values of $\dfrac{d^2y}{dx^2}$ tell you about the graph of y? Consider where the second derivative is positive, where it is negative, and where it equals zero. The terms *concave up* and *concave down* may be useful in your description.

L8.6 Repeat the steps of #L8.5 for the following functions.

- **(a)** $y = \sin x$
- **(b)** $y = x^4 + 2x^3 - 7x^2 - 8x + 12$
- **(c)** $y = x - \dfrac{1}{x}$

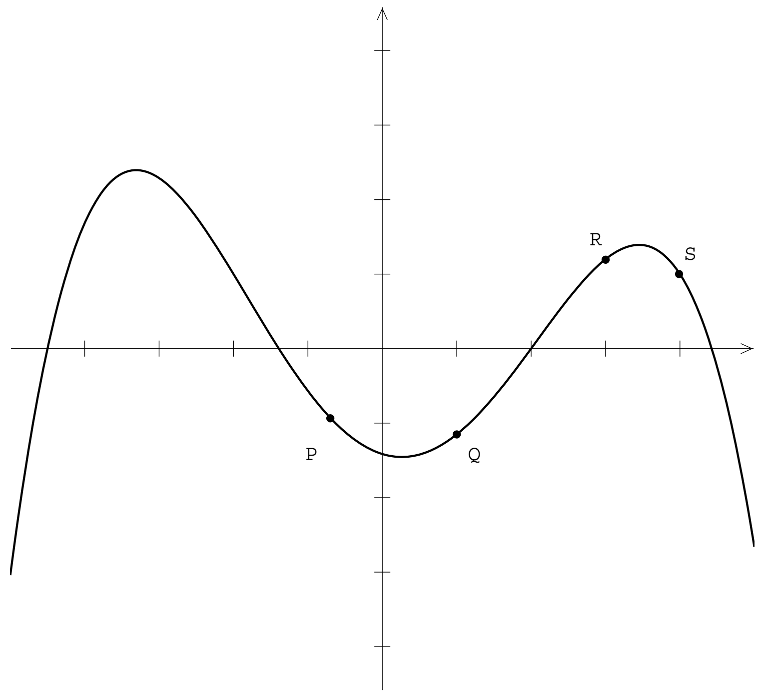
L8.7 Fill in the blanks in the following statements based on your findings.

- **(a)** If the second derivative is positive, then the first derivative is __________ and the graph of the function is __________.
- **(b)** If the second derivative is negative, then the first derivative is __________ and the graph of the function is __________.
- **(c)** If the second derivative changes sign (goes from positive to negative or negative to positive), then the first derivative has a local __________ and the graph of the function changes __________. This location on the function is called a *point of inflection*.

L8.8 Submit a report that summarizes in the form of a table what the first derivative tells you about the graph of a function and what the second derivative tells you about the graph of the first derivative and hence the graph of the function.

P9.1 The figure below shows the graph of $y = f(x)$, where f is a *differentiable* function. The points P, Q, R, and S are on the graph. At each of these points, determine which of the following statements applies:

(a) f' is positive	**(b)** f' is negative	**(c)** f is increasing
(d) f is decreasing	**(e)** f' is increasing	**(f)** f' is decreasing



The graph $y = f(x)$ is called *concave up* at points P and Q, and *concave down* at points R and S.

P9.2 Suppose that $f''(a) = 0$ and that $f''(x)$ changes from negative to positive at $x = a$. What does this tell you about the point $(a, f(a))$ on the graph of $y = f(x)$?

P9.3 Use derivatives to find coordinates for the inflection points on the graph of $y = xe^{-x}$. Examine a graph to confirm your answers.

P9.4 Suppose that $f''(a) = 0$ and that $f''(x)$ changes from positive to negative at $x = a$. What does this tell you about the point $(a, f(a))$ on the graph of $y = f(x)$?

P9.5 The second derivative of $f(x) = x^4$ is 0 when $x = 0$. Does that mean the origin is an inflection point on the graph of f? Explain.

P9.6 A plastic box has a square base and rectangular sides, but no top. The volume is 256 cc. What is the smallest amount of plastic that can be used to make this box? What are the dimensions of the box?

P9.7 (Continuation) The previous problem can be solved graphically without using calculus techniques, and this strategy is often used in Book 3 problems. So why do we need calculus? Suppose you want to solve the problem in general for any volume V, not just $V = 256\ cc$. Solve this new problem using the techniques of calculus. Your minimum surface area and the accompanying dimensions will be given in terms of V.

P9.8 At an inflection point, the tangent line does something unusual, which tangent lines drawn at non-inflection points do not do. What is this unusual behavior?

P9.9 A windmill extracts energy from a stream of air according to the power function $P(x) = 2kx^2(V - x)$, where k is a constant, V is the velocity of the wind, and x is the average of the wind speeds in front and behind the windmill.

- **(a)** Find the derivative of P.
- **(b)** Use the derivative to find the value of x that maximizes the power P. Your answer will be in terms of the parameters k and V. What is the maximum power?
- **(c)** According to Betz's Law, the maximum power that can be extracted from the wind is 16/27 times the power in the wind stream, which is $\frac{k}{2} \cdot V^3$. Verify that the value for P found in (b) is indeed $\frac{16}{27} \cdot \frac{k}{2} \cdot V^3$.

P9.10 Find the extreme points on $y = x + \frac{1}{x}$ by using a derivative. Confirm your answer graphically.

P9.11 (Continuation) Find the extreme points on $y = x + \frac{c}{x}$, where c is a positive constant. Can you confirm your answer graphically? Explain.

P9.12 Suppose that $f'(a) = 0$ and that $f'(x)$ changes from positive to negative at $x = a$. What does this tell you about the point $(a, f(a))$ on the graph of $y = f(x)$?

P9.13 Suppose that $f'(a) = 0$ and that $f'(x)$ changes from negative to positive at $x = a$. What does this tell you about the point $(a, f(a))$ on the graph of $y = f(x)$?

P9.14 Suppose that $f'(a) = 0$ but that $f'(x)$ does not change sign at $x = a$. What does this tell you about the point $(a, f(a))$ on the graph of $y = f(x)$? Show a graph that has this property.

P9.15 Suppose that $f(x)$ is defined for $-4 \leq x \leq 4$, and that its derivative f' is shown below. Use the information in this graph and the additional fact $f(-1) = 3$ to answer the following:

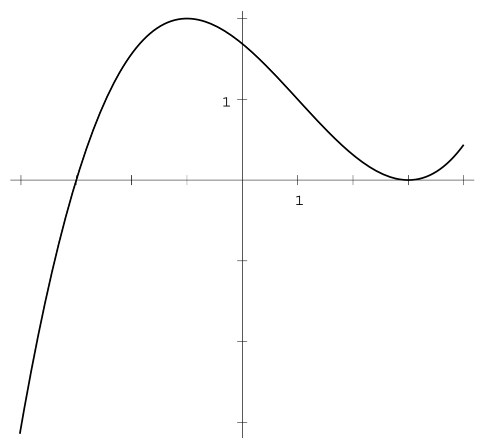
(a) Is it possible that $f(3) \leq 3$? Explain.

(b) Is it possible that $11 \leq f(3)$? Explain.

(c) For what x does $f(x)$ reach its maximum value?

(d) For what x does $f(x)$ reach its minimum value?

(e) Estimate the minimum value and make a sketch of $y = f(x)$ for $-4 \leq x \leq 4$.



The following questions refer to the article from The Economist that follows.

P9.16 In the second paragraph, the author states: "Nobody cares much about inflation; only whether it is going up or down." How can this statement be translated into the language of derivatives?

P9.17 "National debt" refers to how much the government has borrowed throughout the years of its history. A "budget deficit" refers to how much a government spends in a year beyond its annual revenues (which come mainly from taxes).

(a) How is one of these related to the other by differentiation?

(b) Using the language of calculus, explain what happens to national debt if budget deficits are growing each year. What if the budget deficits are declining? What would have to occur with budget deficits for the national debt to decrease?

P9.18 Find three other examples in the article that involve the way things change. Translate your examples into the language of derivatives.

P9.19 What does an increase in the amount of change in a quantity say about the particular phenomenon? What about a decrease in the amount of change?

THE TYRRANY OF DIFFERENTIAL CALCULUS $\frac{d^2P}{dT^2} > 0 = \text{misery}$

from The Economist, 6 April 1991

Rates of change and the pace of decay

"The pace of change slows," said a headline on the *Financial Times's* survey of world paints and coatings last week. Growth has been slowing in various countries — slowing quite quickly in some cases. Employers were invited recently to a conference on Techniques for Improving Performance Enhancement. It's not enough to enhance your performance, Jones, you must improve your enhancement.

Suddenly, everywhere, it is not the rate of change of things that matters, it is the rate of change of rates of change. Nobody cares much about inflation; only whether it is going up or down. Or rather, whether it is going up fast or down fast. *"Inflation drops by a disappointing two points,"* cries the billboard. Which, roughly translated, means that prices are still rising, but less fast than they were, though not quite as much less fast as everybody had hoped.

No respectable American budget director has discussed reducing the national debt for decades; all talk sternly about the need to reduce the budget deficit, which is, after all, roughly the rate at which the national debt is increasing. Indeed, in recent years it is not the absolute size of the deficit that has mattered so much as the trend: is the rate of change of the rate of change of the national debt positive or negative?

Blame Leibniz, who invented calculus (yes, so did Newton, but he called it fluxions, and did his best to make it incomprehensible). Rates of change of rates of change are what mathematicians call second-order differentials or second derivatives. Or blame Herr Daimler. Until the motor car came along, what mattered was speed, a first-order differential. The railway age was an era of speed. With the car came the era of acceleration, a second-order differential: the fact that a car can do 0-60mph in eight seconds is a far more important criterion for the buyer than that it can do 110mph. Acceleration limits are what designers of et fighters, rockets and racing cars are chiefly bound by.

Politicians, too are infected by ever higher orders of political calculus. No longer is it necessary to have a view on abortion or poll taxes. Far better to commission an opinion poll to find out what the electorate's view is, then adopt that view. Political commentators, and smart-ass journalists, make a living out of the second-order differential: predicting or interpreting what politicians think the electorate thinks. Even ethics has become infected (*"You did nothing wrong, Senator, and even the appearance of it does not stink, but people might think it will appear to stink"*).

Bring back the integral

It boils down to biology. There is virtually nothing in the human brain or the nerves that measures a steady state. Everything responds to change. Try putting one hand in hot water and the other in cold water. After a minute put both hands in tepid water. It will feel hot to one, cold to the other. The skin's heat sensors measure changes in temperature.

The visual system of the from is a blank screen on which on things that move (delicious flies or dangerous enemies) show up, like shooting starts in a night sky. Human eyes have "edge detectors" to catch second-order differentials. They respond to the places where the rate of change of the rate of change of light hits zero — these mark the edges of things.

Soon second-order differentials will be passé, and the third order will be all the rage, with headlines reading *"inflation's rate of increase is leveling off"*, or *"growth is slowing quite quickly"*. Frenzy will then be a steady state. It will be high time to reverse the slide into perpetual differentiation. Workers of the world, integrate!

Prelaboratory 9: The Chain Rule for Derivatives

PL9.1 *Composition of functions.* The function h defined by $h(x) = f(g(x))$ is a composition of functions in which f is composed with g. Let $f(x) = \sin x$ and $g(x) = 2\pi x$. Find expressions for each of the following functions:
(a) $h(x) = f(g(x))$ **(b)** $u(x) = g(f(x))$

PL9.2 (Continuation) Find the derivatives of the compositions $h(x)$ and $u(x)$ from #1.

PL9.3 Given the functions $f(x) = e^x$ and $g(x) = 2x - 2$, differentiate the following compositions of these functions.
(a) $y = f(g(x))$ **(b)** $y = g(f(x))$

PL9.4 Find the derivatives of each of the following functions:

(a) $y = e^{x-3}$

(b) $y = e^{2x}$

(c) $y = e^{-x^2}$

(d) Test your answer to the previous part using graphing technology. Notice that (a) and (b) are straightforward derivatives, but the (c) is not even though the two functions composed, e^x and $-x^2$, have familiar derivatives.

PL9.5 Find the derivatives of each of the following functions:

(a) $y = \dfrac{1}{x}$

(b) $y = \dfrac{1}{x^2}$

(c) $y = \dfrac{1}{x^2 - 1}$

(d) Test your answer to part (c) using graphing technology. Notice that (a) and (b) are straightforward derivatives, but the (c) is not even though the two functions composed, $1/x$ and $x^2 - 1$, have familiar derivatives.

PL9.6 The following functions are each the composition of two basic functions that you're likely more familiar with. Name the two functions for each composition.

(a) $h(x) = \dfrac{1}{\sin(x)}$

(b) $u(x) = (x^2 + 5)^2$

(c) $v(x) = \sqrt{25 - x^2}$

Laboratory 9: The Chain Rule for Derivatives

L9.1 This lab investigation is all about how to find the derivative of a composition of functions. In the prelab, notice that we could find derivatives of compositions where one of the functions is linear. (We looked at this in Lab: Transformations.) Use your knowledge of derivatives of functions with transformations to find the derivatives of the following functions.

(a) $y = e^{2x-6}$

(b) $y = (3-x)^3$

(c) $y = \sin(2\pi(x - \pi/4))$

(d) $y = \ln(4x - 4)$

L9.2 There is a pattern to the derivatives in L9.1: take the derivative of the "outer" function (exponential, cubic, sine, natural log) evaluated at the inner function (which is linear), and multiply the result by the slope of the inner function. In the case of L9.1(a), this leads to the following steps:

Outer function	e^x
Derivative of outer function	e^x
. . . Evaluated at inner function	e^{2x-6}
And multiplied by slope of inner	$2e^{2x-6}$

Now apply the same pattern to a composition with a non-linear function e^{-x^2}:

Outer function	e^x
Derivative of outer function	e^x
. . . Evaluated at inner function	e^{-x^2}
And multiplied by slope of inner	$-2xe^{-x^2}$

Confirm graphically that this is the actual derivative. Now find the derivatives of the following compositions:

(a) $h(x) = \dfrac{1}{\sin(x)}$

(b) $u(x) = (x^2 + 5)^2$

(c) $v(x) = \sqrt{25 - x^2}$

L9.3 In L2 we showed that for $h(x) = e^{-x^2}$, the derivative is $h'(x) = -2xe^{-x^2}$. If we write h as the composition $h(x) = f(g(x))$ with $f(x) = e^x$ and $g(x) = -x^2$, then we can write the derivative as $h'(x) = f'(g(x))g'(x)$.(Confirm that this is a true statement.) This is an example of the Chain Rule for derivatives. Find the derivative of $h(x) = (\cos(x))^2$ using the *Chain Rule*.

L9.4 Find the derivative of $(x^2 - 1)^2$ by two different methods:

- **(a)** multiply $x^2 - 1$ by itself, collect like terms, then differentiate term-by-term;
- **(b)** use the Chain Rule.

Show that both answers are equivalent expressions.

L9.5 Another way of understanding the Chain Rule is through linear approximation. We begin with the composite function $h(x) = f(g(x))$.

- **(a)** Using the linear approximation for $g(x)$ near $x = a$, show that $h(x) \approx f(g'(a)(x - a) + g(a))$.
- **(b)** Confirm that $\dfrac{d}{dx}[f(mx + b)] = f'(mx + b) \cdot m$ by referring to your previous work with derivatives of functions composed with linear functions.
- **(c)** Hence show that $h'(x) \approx f'(g'(a)(x - a) + g(a)) \cdot g'(a)$, and thus explain the following equation: $h'(a) \approx f'(g(a)) \cdot g'(a)$.
- **(d)** How does (c) establish the Chain Rule $h'(x) = f'(g(x))g'(x)$?

L9.6 Write a paragraph explaining your understanding of the chain rule. Be sure to state the chain rule in words and in symbols. Include, with examples worked out, what types of functions you can now find the derivatives of that you were unable to calculate previously.

P10.1 The following work is a formal proof of the Chain Rule. Annotate the proof with comments explaining each step in the proof.

Let $y = f(g(x))$ where f and g are continuous, differentiable functions for all x in their domains.

(a) $y' = \lim_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{h}$

(b) $y' = \lim_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{h} \cdot \dfrac{g(x+h) - g(x)}{g(x+h) - g(x)}$

(c) $y' = \lim_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \dfrac{g(x+h) - g(x)}{h}$

(d) $y' = \lim_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h}$

(e) $y' = \lim_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot g'(x)$

Let $k = g(x+h) - g(x)$. Notice that this can be rewritten as $g(x+h) = g(x) + k$, and that as $h \to 0$, $k \to 0$.

It is also true that $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$. So now let $a = g(x)$:

(f) $y' = \lim_{k\to 0} \dfrac{f(g(x)+k) - f(g(x))}{g(x+h) - g(x)} \cdot g'(x)$

(g) $y' = \lim_{k\to 0} \dfrac{f(g(x)+k) - f(g(x))}{k} \cdot g'(x)$

(h) $y' = f'(g(x)) \cdot g'(x)$

P10.2 The function $y = e^{-x^2}$ is widely applied in statistics by using various transformations of the graph, leading to what is known as the normal curve.

(a) Describe the shape of this graph. What are the main features of the graph?

(b) Find the first and second derivative of this function.

(c) Use graphs and algebra to explore the concavity of this function. Find the coordinates of any points of inflection, the points where the concavity changes.

P10.3 Find the derivatives of the following functions.

(a) $H(u) = (\sin u)^3$

(b) $C(t) = \dfrac{e^t + e^{-t}}{2}$

(c) $v(x) = \sqrt{r^2 - x^2}$

(d) $u = (t+1)^2 \ln t$

P10.4 The annual government expenditures E in Geerland can be modeled as a function of population P by $E(P) = 10\sqrt{P}$, with units of thousands of dollars. The population is growing and is modeled by the function $P(t) = 100e^{0.03t}$, where t is in years.

(a) Express E as a function of time.

(b) Find the derivative of E with respect to time.

(c) Comment on the graphs of $E(t)$ and $E'(t)$.

P10.5 (Continuation) Suppose now the population of Geerland grows according to the function $P(t) = 10000 - 9900e^{-0.03t}$.

(a) Graph P over a suitable domain. Why do you think this model is called *constrained growth*? What is the constraint, or upper limit, on the population?

(b) Find the derivative of E with respect to time using the constrained growth model for population.

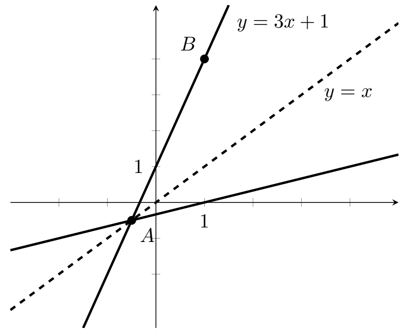
(c) Comment on the graphs of E and the derivative of E.

P10.6 If we ignore the effects of air resistance, a falling object has a constant acceleration of 9.8 m/sec^2.

(a) If the object falls from an initial height of 100 meters with initial velocity of 0, find equations for the velocity and height as functions of time (until the object hits the ground). With what speed does it strike the ground?

(b) If the object falls from an initial height of 100 meters and it is initially projected upward with a velocity of 20 m/sec, find equations for the velocity and height as functions of time (until the object hits the ground). What is the maximum height and when does that occur? With what speed does it strike the ground?

P10.7 The graph shows a line and its reflection across the line $y = x$. Find the images of points A and B, then find the equation of the image line. What is the relationship between the slope of the original line and the slope of its reflection?

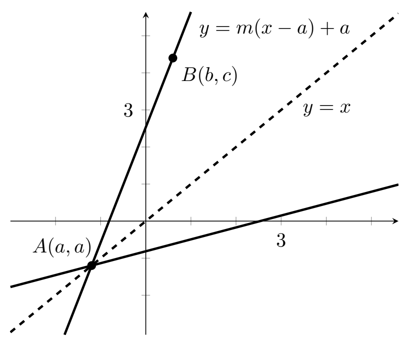


P10.8 (Continuation) The graph shows $y = m(x - a) + a$ and its reflection over $y = x$.

(a) Explain why $y = m(x - a) + a$ fits the line shown.

(b) Find the images of A and B, and the equation of the reflected line.

(c) What is the relationship between the slope of a line l and the slope of the reflection of l over $y = x$?



P10.9 (Continuation) If the set of points (x, y) that satisfy the equation $y = x^2$ is reflected through the line $y = x$, then the resulting set of points satisfies which of the following equations?

(a) $x = y^2$

(b) $y = \sqrt{x}$

P10.10 *Implicit Differentiation.* Find the derivative of both sides of the equation for the unit circle equation $x^2 + y^2 = 1$. Term-by-term differentiation is straightforward for two of the three terms of this equation, specifically $\dfrac{d}{dx}\left(x^2\right) = 2x$ and $\dfrac{d}{dx}(1) = 0$.

(a) Explain how the chain rule is used in the differentiation $\dfrac{d}{dx}\left(y^2\right) = 2y \cdot \dfrac{dy}{dx}$. This derivative assumes that y is an *implicitly defined function of* x.

(b) Put all the pieces together and solve for $\dfrac{dy}{dx}$ from the circle equation. How else could you have found this derivative?

P10.11 Find the derivatives of the following functions.

(a) $y = \dfrac{1}{\sin x}$

(b) $u = (t - 1)^3 \ln t$

(c) $f(x) = e^{\cos x}$

P10.12 In radian mode, the parametric equation $(x, y) = (5\cos t, 15\sin t)$ traces an ellipse as t varies from 0 to 2π.

(a) Confirm by graphing and by algebra that this ellipse also has equation $9x^2 + y^2 = 225$.

(b) Find the t-value that corresponds to the point $(4, 9)$.

(c) Find the components of velocity $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$, and find their values at $(4, 9)$.

(d) On a graph of the ellipse, show the vector $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$ with its tail at $(4, 9)$ using the values of the components found in (c). Use these components of velocity to find the slope of the line tangent to the curve at $(4, 9)$.

P10.13 You previously discovered that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$. This can also be established through the derivative of e^x, which is the inverse of $\ln x$:

(a) starting with $y = \ln x$, write x as a function of y, so that $e^y = x$;

(b) use implicit differentiation to show that $e^y \dfrac{dy}{dx} = 1$;

(c) solve for $\dfrac{dy}{dx}$ and substitute for y in terms of x to show that $\dfrac{dy}{dx} = \dfrac{1}{x}$.

P10.14 Explain why the graphs of inverse functions are related by a reflection across the line $y = x$.

(a) On the same system of coordinate axes, and using the same scale on both axes, make careful graphs of both $y = \ln x$ and $y = e^x$.

(b) Let $P = (a, b)$ be a point on the graph of $y = \ln x$, and let $Q = (b, a)$ be the corresponding point on the graph of $y = e^x$. How are the slopes of the curves related at these two points? Explain.

(c) Find the slopes at P and Q by using the derivatives of $\ln x$ and e^x. Confirm the relationship between the slopes of corresponding points on the graphs of these inverse functions by letting $a = 0.5$, 1, and 1.5.

P10.15 If $y = f(u)$ and $u = g(x)$, then the Chain Rule can also be written as $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

(a) How does this notation compare with $h'(x) = f'(g(x))g'(x)$? Explain how the two forms are the same.

(b) For the function $y = e^{\sin x}$, identify the functions f and g that are composed with f being the "outer" function and g being the "inner" function.

(c) Find $\dfrac{dy}{dx}$ using the Chain Rule.

Prelaboratory 10: Discovering Differential Equations

PL10.1 The number of people becoming zombies during the apocalypse can be modeled according to the linear equation $z = 3t + 1$, where z is the number of zombies in the population and time t is measured in days.

(a) When $t = 0$, how many zombies are present in the population? How do you know?

(b) Write an equation for the change in number of zombies $\frac{dz}{dt}$ at time t in the form $\frac{dz}{dt} =$ __________.

(c) Determine another linear relationship between the number of zombies in the population and time, in which the growth rate at time t is the same as in (b). How have you transformed the graph of $z = 3t + 1$?

(d) The equation in (b) represents all linear functions whose rate of change is a constant 3 zombies per day. This is an example of a *differential equation.* Now write a differential equation for the rate of change of a general linear function $y = mx + b$. How many functions does your differential equation represent?

(e) Solve the differential equation $\frac{dy}{dx} = 10$ for the particular solution that contains the point $(0, 5)$.

PL10.2 Examining your work from the previous problem, in what ways is this model realistic? In what ways is it not? What would you have to adjust in order to make your model more accurate?

Lab 10: Discovering Differential Equations

As we have increased the complexity of our study of the zombie apocalypse, it has become clear that having the following functions would be very helpful, but is not always possible:

- $Z(t)$ the number of zombies in a population on a particular day
- $H(t)$ the number of healthy, non-zombies in a population on a particular day
- $I(t)$ the number of infected individuals on a particular day

In this lab, we will formalize our understanding of how we can work with these functions and their derivatives to build more accurate models.

L10.1 For each of the following differential equations, explain the meaning in your own words. Consider what the differential equation is a function of. Be as specific as possible.

(a) $\frac{dZ}{dt} = 10$

(b) $\frac{dH}{dt} = -3t + 100$

(c) $\frac{dI}{dt} = 0.1 * H(t)$

(d) $\frac{dZ}{dt} = 2Z(t) * (1 - \frac{Z(t)}{1000})$

L10.2 Consider if we could say $H(t) = 4123e^{-0.02t}$, where t is measured in days since the first infection.

(a) Determine the initial population of healthy people and the initial instantaneous growth rate.

(b) Write a differential equation for the growth rate of H at any time t. Notice that within this differential equation, you will find H multiplied by some constant. Rewrite the differential equation in terms of H.

(c) Determine another exponential relationship in which the differential equation, in terms of H, is the same as in (b). How have you transformed the graph of $H = 4123e^{-0.02t}$?

(d) Write a differential equation for the rate of change of a general exponential function with base e, initial value A, and growth rate k.

(e) Solve the differential equation $\frac{dH}{dt} = 0.15H$ for the particular solution with an initial value of $H = 225$.

L10.3 *Newton's Law of Cooling* describes how an object's temperature changes depending on its starting temperature and the ambient temperature around it. Imagine that you have a cup of coffee on a cold winter's day (the coffee starts at 95 degrees Celsius and the ambient temperature is 5 degrees Celsius), whose temperature is given by $C(t)$, where t is the number of minutes after the coffee is poured.

(a) Newton's Law of Cooling states that the rate at which an object changes temperature is proportional to the difference between its temperature and that of its surroundings. Is this intuitive? Explain.

(b) Explain why $\frac{dC}{dt} = -k(C - 5)$ (k is a positive constant) describes the rate of change of the temperature of the beverage. Why is there a minus sign in the equation?

(c) What factors might impact the size of the k value?

(d) If $k = 0.1$, write the differential equation describing the rate of change of the coffee's temperature and sketch its graph.

L10.4 Throughout this lab, you have considered linear, quadratic, and exponential functions. For which of these equations is the rate of change at time t constant? For which of these equations is the percent rate of change at time t constant?

L10.5 Write a paragraph summarizing what you have learned in this lab about differential equations. Be sure to include observations about *families of functions* that have the same differential equation. You will also want to incorporate the concepts of constant change, linear change, and constant percentage change in your write-up.

P11.1 Find $\dfrac{dy}{dx}$ for each of the following functions:

(a) $y = x\sqrt{1-x^2}$ **(b)** $y = \sin(x^2)$ **(c)** $y = 9 - \cos^2(90x)$

P11.2 The core temperature of a potato that has been baking in a 375° oven for t minutes is modeled by the equation $C = 375 - 300(0.96)^t$. Find a formula for the rate of change of the temperature in degrees per minute for this potato.

P11.3 The equations $y = 2^x$ and $y = \log_2 x$ represent inverse functions. Find the derivatives of each function, and explain how they are related.

P11.4 A can with a volume of 1000 cc consists of a cylinder with a top and a bottom. The height of the can is h cm, and the radius of the top and bottom is r cm.

(a) What is the surface area of the side? What is the surface area of the top and bottom? Write an equation for the total surface area A in terms of r and h.

(b) Explain why $\pi r^2 h = 1000$. Use this equation to express surface area A as a function of r alone.

(c) Use a derivative to find the value of r that gives the minimum value of A. Confirm your answer graphically. What is the value of h that corresponds to this r-value? How would you describe the shape of this can of minimum surface area?

P11.5 Given the graph of a set of points (a, b), the set of points (b, a) is a reflection over what line?

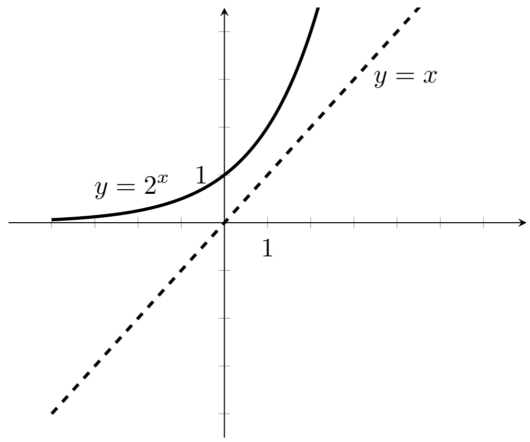
P11.6 With the help of the Chain Rule and the Power Rule, write out the derivative of $f(x) = (\sin x)^{1/2}$. Then find a formula for the general example of this type, which has the form $f(x) = (g(x))^n$.

P11.7 Using the Chain Rule and Power Rule, show that the derivative of $R(x) = \dfrac{1}{g(x)}$ is $R'(t) = -\dfrac{g'(t)}{[g(t)]^2}$. (Hint: first write R as a power function $R(x) = [g(x)]^{-1}$).

P11.8 (Continuation) *Quotient Rule.* A function Q defined as a ratio of two functions $Q(x) = \dfrac{f(x)}{g(x)}$ can be differentiated by first rewriting Q as a product $Q(x) = f(x) \cdot \dfrac{1}{g(x)}$. Show that $Q'(x)$ can be written as:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

P11.9 Sketch the graph of the reflection of $y = 2^x$ over $y = x$ on the axes below.



P11.10 Graph the function $f(x) = x - \sin x$ and its inverse. Although one cannot find an explicit formula for $f^{-1}(x)$, one can find its derivative at certain points.

(a) Find the derivative of $f^{-1}(x)$ at the point (π, π).

(b) Find the derivative of $f^{-1}(x)$ at the point $(\frac{\pi}{2} - 1, \frac{\pi}{2})$.

P11.11 Use implicit differentiation to find the slope of the ellipse $x^2 + 9y^2 = 225$ at $(9, 4)$. In other words, find $\dfrac{dy}{dx}$ without first solving for y as an explicit function of x.

P11.12 Let $f(x) = x^3$, $g(x) = x^4$, and $k(x) = x^7$. Notice that $k(x) = f(x) \cdot g(x)$. Is it also true that $k'(x) = f'(x) \cdot g'(x)$? Explain.

P11.13 Use a derivative to calculate the slope of the curve $y = x^3$ at the point $(2, 8)$. Now find the slope of the curve $x = y^3$ at the point $(8, 2)$. How could you have found the second slope from the first slope without finding another derivative?

P11.14 After first writing $\tan x = \dfrac{\sin x}{\cos x}$, use the Quotient Rule to show that $\dfrac{d}{dx}(\tan x) = \sec^2 x$. Recall that $\sec x = \dfrac{1}{\cos x}$.

P11.15 Find the derivatives of the following:

(a) $f(x) = x^3 + 3^x$

(b) $M(\theta) = 8\tan(3\theta)$

(c) $H(u) = (\sin u)^3$

P11.16 Find $\dfrac{dy}{dx}$ for each of the following:

(a) $y = \dfrac{x-1}{x+1}$ **(b)** $y = \dfrac{1}{\cos x}$ **(c)** $y = \dfrac{x}{x^2+1}$

P11.17 In radians, the parametric equation $(x, y) = (4\cos t, 3\sin t)$ traces an ellipse as t varies from 0 to 2π.

(a) Confirm by graphing and by algebra that this ellipse also has equation $9x^2 + 16y^2 = 144$. You will need the identity $\cos^2 t + \sin^2 t = 1$.

(b) Find the t-value that corresponds to the point $(2, \frac{3\sqrt{3}}{2})$.

(c) Find the components of velocity $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and find their values at $(2, \frac{3\sqrt{3}}{2})$.

(d) On a graph of the ellipse, show the vector $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$ with its tail at $(2, \frac{3\sqrt{3}}{2})$ using the values of the components found above. Use these components of velocity to find the slope of the line tangent to the curve at $(2, \frac{3\sqrt{3}}{2})$.

P11.18 The equation $A(t) = 6.5 - \frac{20.4t}{t^2 + 36}$ models the pH of saliva in the mouth t minutes after eating candy. The lower the pH, the more acidic is the saliva. A pH of 7 is neutral.

(a) Using a graph of this function, what is the normal pH in the mouth? How long after eating candy does it take for the pH to return to within 0.1 of normal?

(b) After eating the candy, when is the mouth becoming more acidic? When is the acidity decreasing?

(c) At what time is there a point of inflection on the graph? What interpretation can you give to this point in the context of the problem?

P11.19 Consider the curve, in radian mode, defined by $(x, y) = (\cos t - 3\sin t, 2\cos t \sin t)$.

(a) Graph this curve for $0 \le t \le 2\pi$. In which direction is the curve swept out as t increases?

(b) For what points on the curve is $\frac{dy}{dt} = 0$? What are the equations of the tangent lines at those points?

(c) For what points on the curve is $\frac{dx}{dt} = 0$? What are the equations of the tangent lines at those points?

(d) Find the slope of the curve at any point in terms of the parameter t. Use this expression to find the equation of the tangent line at the point with $t = 1$.

P11.20 The acceleration due to gravity on the Moon is about 1.6 m/sec^2.

(a) Represent the acceleration with a differential equation using the derivative of the velocity v of a falling object on the Moon.

(b) Use the differential equation from (a) to find an equation for the velocity of an object that is propelled upward with an initial velocity of 25 m/sec.

(c) Since $v = \dfrac{dh}{dt}$, where h is the height of the object, you can use the result of (b) to write a differential equation for the height of the object. Solve this for h as a function of t given that the initial height of the object is 3 meters.

P11.21 Suppose the acceleration due to gravity is a constant g m/sec^2, the initial velocity is v_0 m/sec, and the initial height is h_0 meters. As in the previous problem, write a differential equation for acceleration, then use this to find the differential equation for velocity. Finally, solve it for velocity to get an equation for h as a function of t. You should obtain this well-known formula from physics: $h = -\dfrac{g}{2}t^2 + v_0 t + h_0$.

Prelaboratory 11: Projectile Motion

PL11.1 A vector of length 4 makes an angle of 30° with the x-axis. What are the x- and y-components of this vector?

PL11.2 An object has a velocity vector with horizontal component 12 and vertical component 16. What is the speed of the object?

PL11.3 What are the components of a velocity vector v that makes an angle θ with the x-axis?

PL11.4 A projectile flies across a plain with an acceleration vector [0, -9.8]. We ignore air resistance and the only force is gravity, which is in the vertical direction; hence, the x-component of acceleration is 0 and the y-component is $-9.8\ \text{m/sec}^2$ (the acceleration due to gravity). The initial velocity of the projectile is 5 m/sec at a 45° angle.

(a) What are the x- and y-components of the initial velocity vector?

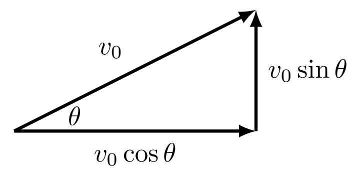
(b) Confirm that the velocity vector as a function of time t is $[5\cos 45°, -9.8t + 5\sin 45°]$.

(c) The projectile is launched from an overlook 10 m above the plain. Confirm that the position vector as a function of time is $[(5\cos 45°)t, -\frac{9.8}{2}t^2 + (5\sin 45°)t + 10]$.

(d) When will the projectile land? Where will it land? What will be the speed at the moment of impact?

Laboratory 11: Projectile Motion

L11.1 You have seen that trigonometry helps decompose a vector into its vertical and horizontal components. The diagram shows how to decompose a projectile vector, v_0, that makes an angle of θ with the ground.



(a) Explain how the three vectors are related. How are their lengths related?

(b) If we assume that gravity acts with a force of g, that we are ignoring air resistance in our model, that the projectile is launched from an initial height of h_0 and additionally that there is no air resistance, we can use the component vectors to build parametric equations that describe the motion of the projectile. Convince yourself that the following parametric equations describe such a setup.

$$x(t) = (v_0 \cos\theta)\, t \qquad y(t) = -\frac{g}{2}t^2 + (v_0 \sin\theta)\, t + h_0$$

(c) Throughout this lab we will use the acceleration due to gravity $g = 9.81$ m/sec^2. Suppose we let $\theta = 40°$, the initial velocity $v_0 = 50$ m/sec, and the initial height $h_0 = 10$ m. Write out the equations for $x(t)$ and $y(t)$.

(d) Use technology to graph the path of the projectile.

(e) How long does it take for the projectile to hit the ground? Where does it hit the ground? What is the speed at the moment of impact?

L11.2 Open the Desmos simulation in the file Projectile Motion - Storming the Castle, which contains a catapult and a castle. Your catapult always launches the projectile with an initial velocity of 50 m/sec, but you can adjust the angle, a, of the launch. Launching is accomplished by clicking on the animation arrow next to the slider, b, for time.

(a) Ignore the castle for now and try to land the projectile as far away as you can as measured by the x-coordinate of where it crosses the x-axis.

(b) For which angle can you reach the furthest point on the plain? Ignore the castle for now.

(c) For which angle will the projectile land on top of the closest tower? Is there another angle that will also work? Make sure your answer references the location and height of the tower.

L11.3 Return to the Desmos simulation mentioned above to answer the following.

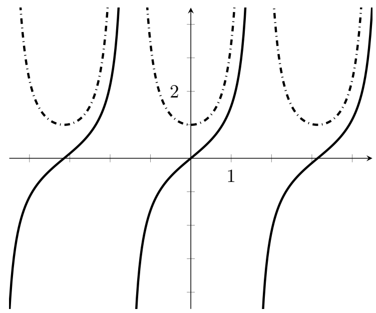
(a) Write out the velocity components of the projectile when it strikes the tower for each of the two solutions found in #L11.2(c).

(b) What are the speeds with which the projectile strikes the tower in each of the two solutions? Is there an advantage to one launch angle over the other?

(c) Compare the slopes of the trajectories found in #L11.2(c) at the moment of impact on the tower. Use the slopes to find the angles of impact.

L11.4 Compile a report with your results from this lab. Be sure to explain the distinction between speed at impact and angle of impact. What surprising results did you obtain? What problems can you now model that you could not solve prior to this lab?

P12.1 Two functions are shown on the graph below, one of which is the derivative of the other. Which is the derivative function? Write the equations for the two functions.



P12.2 Solve the following *antiderivative* questions. In other words, given their derivatives below, find W, F, and g. How can you check your answers? Could there be more than one answer for each?

(a) $\dfrac{dW}{dx} = 10x^9$ **(b)** $F'(u) = 120e^{0.25u}$ **(c)** $g'(t) = -10\pi \sin 2\pi t$

P12.3 What are the dimensions of the largest cylinder that can be inscribed in a sphere of radius 1? You solved a problem like this in precalculus by writing the volume of the cylinder in terms of its radius alone (or its height), then finding the maximum point on the volume graph. Now solve the problem using the techniques of calculus. Verify your answer graphically.

P12.4 (Continuation) What are the dimensions of the largest cylinder that can be inscribed in a sphere of radius R? Your answer should be in terms of the parameter R. This parameter makes the problem difficult to solve graphically, hence techniques of calculus are useful.

P12.5 A population grows according to the exponential function $P(t) = 100e^{0.07t}$.

(a) What is the initial population? What is the population after one year? What is the average percent growth over the first year?

(b) What is $P'(t)$, the instantaneous growth rate? What is the initial instantaneous growth rate? How does this compare with the average percent growth over the first year? Explain why there is a difference between these two values.

(c) The function $P(t)$ can also be expressed as $100\,(1+r)^t$ by substituting $1+r$ for $e^{0.07}$. What is the value of r? How is this related to (a) and (b)?

P12.6 The slope of the curve $y = \tan x$ is equal to 2 at exactly one point P whose x-coordinate is between 0 and $\frac{\pi}{2}$. Let Q be the point where the curve $y = \tan^{-1} x$ has slope $\frac{1}{2}$. Find coordinates for both P and Q. How are these coordinate pairs related?

P12.7 A potato, initially at room temperature (70°F), is placed in a hot oven (350°F) for 30 minutes. After being taken out of the oven, the potato sits undisturbed for 30 more minutes on a plate in the same room (70°F). Let $T(t)$ be the temperature of the potato at time t during the 60-minute interval $0 \le t \le 60$. Draw plausible graphs of both $T(t)$ and $T'(t)$. Other than $T(0)$, you are not expected to know any specific values of T.

P12.8 Use implicit differentiation to find the slope of the curve $e^y + x^2 = 2$ at the point $(-1, 0)$. Graph the curve and the tangent line at this point.

P12.9 Alex is in the desert in a jeep, 10 km from the nearest point N on a long, straight road. On the road, the jeep can do 50 kph, but in the desert sands, it can manage only 30 kph. Alex is very thirsty and knows that there is a gas station that has ice-cold Pepsi, located at point P that is 20 km down the road from N. Alex decides to drive there by following a straight path through the desert to a point that is between N and P, and x km from N. The total time $T(x)$ for the drive to the gas station is a function of this quantity x. Find an explicit expression for $T(x)$, then calculate $T'(x)$. Use algebra to find the minimum value of $T(x)$ and the value of x that produces it.

P12.10 Find the derivatives of the following functions.
(a) $u = \dfrac{\cos t}{t}$ **(b)** $H(x) = \dfrac{2x - 1}{x^3}$

P12.11 Given the function $f(x) = \dfrac{\ln x}{x}$, use first and second derivatives to find the coordinates of all maximum and minimum points, and any points of inflection. Use a graph to verify your answers.

P12.12 If the slope of the graph $y = f(x)$ is m at the point (a, b), then what is the slope of $x = f(y)$ at the point (b, a)?

P12.13 Find the derivatives of the following functions.
(a) $v = e^{-x} \sin x$ **(b)** $g(t) = \dfrac{t}{1 + t}$ **(c)** $y = \ln(\sin(x))$

P12.14 Find the value of $\lim_{h\to 0} \dfrac{\ln(a+h)-\ln(a)}{h}$ in terms of a without using a calculator.

P12.15 The differential equation $\dfrac{dy}{dt} = -\dfrac{1}{3}y^2$ is given.

(a) Show that $y = \dfrac{3}{t}$ is a solution, but ...

(b) ...that neither $y = \dfrac{3}{t} + 2$ nor $y = \dfrac{2}{t}$ is a solution.

(c) Find another solution to the differential equation.

P12.16 After investigating the derivative of x^n when n is a positive integer and determining that $\dfrac{d}{dx}(x^n) = nx^{n-1}$, we extended this Power Rule to all powers. We now have the tools to examine this extension more critically.

(a) (Negative powers) If the power is a negative integer, we can rewrite the function as $y = \dfrac{1}{x^n}$, where n is a positive integer. Use the Quotient Rule to find $\dfrac{dy}{dx}$, then use the Power Rule to find the derivative of x^{-n}. Recall that $x^{-n} = \dfrac{1}{x^n}$. Your answers should agree.

(b) (Rational powers) Suppose the power is a rational number, so that $y = x^{a/b}$, where a and b are positive integers. We can rewrite the equation as $y^b = x^a$, where y is now an implicit function of x. Use implicit differentiation to solve for $\dfrac{dy}{dx}$, then use the Power Rule to find the derivative of $x^{a/b}$. Your answers should agree.

P12.17 Identify each of the following rules for derivatives.

(a) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx}+v\dfrac{du}{dx}$ **(b)** $\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$ **(c)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Prelaboratory 12: Introducing Slope Fields

PL12.1 Solve the following antiderivative problems by finding the functions with the derivatives given. Could there be more than one answer for each? What does a graph look like if all the solutions are shown?

(a) $dy/dx = 3x^2$

(b) $f'(t) = 4\pi\cos(2\pi t)$

PL12.2 A slope field for a differential equation is a graph in which various points in the coordinate plane are assigned values equal to the derivative at that point, which is then shown as a short line segment with slope equal to the derivative. For example, a slope field for the differential equation $dy/dx = \frac{1}{2}x$ has a segment of slope 0 at the point $(0,0)$ and a segment of slope 1 at the point $(2,1)$.

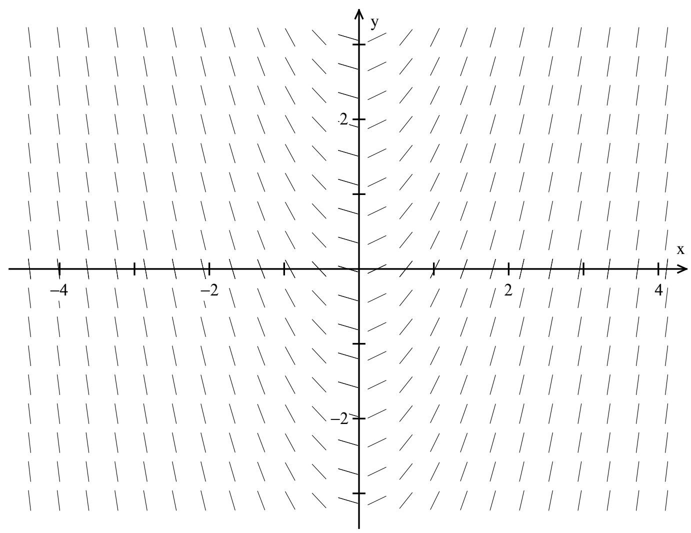
(a) Make a table of the values of the derivative $dy/dx = \frac{1}{2}x$ for the grid of lattice points for $-2 \leq x \leq 2, -2 \leq y \leq 2$.

(b) Draw a slope field of line segments for the results of part (a). Each line segment should be about half a unit long.

(c) Draw a solution curve containing the point $(0,0)$ by sketching a curve through $(0,0)$ that follows the direction of slope field, which is determined by the line segments shown in your graph. Should a solution curve cross over any line segments? What is the equation for this solution curve?

(d) Draw 2 more solution curves, one containing the point $(0,-1.5)$ and the other containing the point $(1,1)$. How are these 3 solution curves related?

The figure below shows the *slope field* for the differential equation $\dfrac{dy}{dx} = 2x$. Each point is assigned a slope equal to the value of the derivative at that point. That slope is represented as a line segment. For example, the point $(1, 1)$ has a segment with slope 2 because $\dfrac{dy}{dx}$ is $2(1) = 2$ at $x = 2$, and similarly the point $(-2, -2)$ has a segment with slope -4. Many other points are represented in the slope field, which allows you to visualize the behavior of the solutions to the differential equation. Since the derivative depends only on x, all of the slopes along vertical lines are the same. A solution curve can be drawn on this graph by choosing a point on the curve and then sketching the curve by following the direction of the slope field, which is why it is also called a *direction field*.



Laboratory 12: Introducing Slope Fields

L12.1 Slope fields can be drawn using Geogebra with the command $SlopeField\left[\dfrac{dy}{dx}, 20\right]$. The number 20 here represents the number of line segments shown, this can be adjusted up or down to get a better picture.

(a) Produce the slope field for $\dfrac{dy}{dx} = 2x$ by entering $SlopeField\,[2x, 20]$. Observe how the slope field changes as you change the graphing window.

(b) Discuss the solution curve that contains the point $(0, 0)$. Use Geogebra to draw the solution curve with the command $SolveODE\,[2x, (0, 0)]$. Is it what you expected?

(c) Discuss the solution curve that contains the point $(-2, -2)$ and draw the curve with Geogebra.

(d) Add 2 more solution curves to your graphing window. What characteristics are shared by all of the solution curves? Do all the curves have the same general behavior?

(e) Use your knowledge of derivatives to write a function $y = f(x)$ for the general solution to the differential equation $\dfrac{dy}{dx} = 2x$. How does the general solution compare with the curves drawn in the slope field?

L12.2 Consider the differential equation $\dfrac{dy}{dx} = y$.

(a) Create a slope field for this differential equation with Geogebra using the command SlopeField [y, 20]. Discuss the appearance of the slope field. In what direction can you move without changing the slopes? Why?

(b) Draw a solution curve containing the point (0, 1) in Geogebra with the command SolveODE [y, (0, 1)]. Draw a different solution curve containing the point (0,-2) in Geogebra with the command SolveODE [y, (0,-2)]. How do the behaviors of these curves differ? Is there any other type of behavior that a solution curve can have?

(c) Solve the differential equation for an explicit function by using your knowledge of derivatives. How does your symbolic solution give you insight into the behavior of solution curves drawn in the slope field?

L12.3 Consider the differential equation $\dfrac{dy}{dx} = \sin(x)$. Create a graph of the slope field, draw some solution curves, and discuss the different behaviors that the solution curves can possess. You should also solve the differential equation and compare your symbolic solution with your observations about the slope field.

L12.4 One version of Newton's law of cooling states that a hot liquid will cool at a rate that is proportional to the difference between the temperature of the liquid and the ambient temperature of the surroundings (such as room temperature, or the temperature inside a refrigerator). This description of change can be written as the differential equation $\frac{dT}{dt} = k(T - A)$, where T is the temperature of the liquid, t is time, k is a constant of proportionality, and A is the ambient temperature.

(a) Let $A = 68°$ and $k = -0.05$. Create a graph of the slope field with the command SlopeField [-0.05(y-68), 20]. Be sure that your window spans the horizontal line $y = 68$. Why?

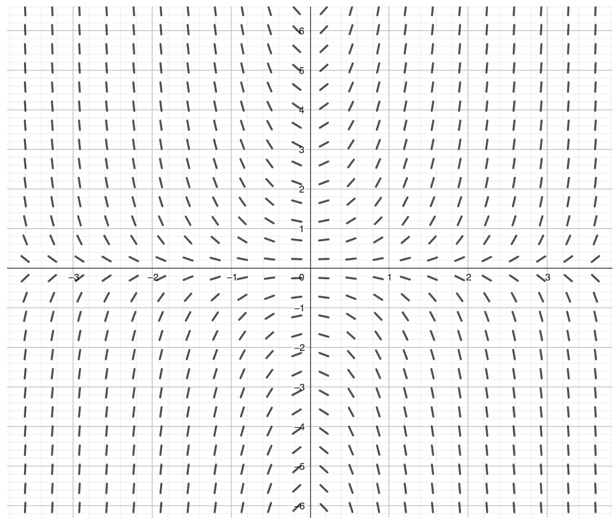
(b) Draw the solution curve with an initial temperature (at time $t = 0$) of 200°by using the command SolveODE [-0.05(y-68), (0, 200)], then, then compare this curve with the solution curve for an initial temperature of 0°. If necessary change your window. What happens if the initial temperature is 68°?

(c) This differential equation is not one that we can solve symbolically at this time; however, you can learn a lot about the solution through the visualization provided by the slope field. Describe the three qualitatively different solutions and how they depend upon the initial temperature.

L12.5 What is a slope field? What does it tell us? How does a slope field provide a visualization of the qualitatively different behaviors of the solutions to a differential equation (using L13.4 as an example)?

L12.6 Turn in a report of this lab along with pictures of the 4 different slope fields with representative solutions you investigated in this lab. (You can cut-and-paste the graphs of slope fields from Geogebra.)

P13.1 Using the slope field below, sketch in the solution curve that passes through the point $(0, 3)$



P13.2 There are many functions for which $f(3) = 4$ and $f'(3) = -2$. A quadratic example is $f(x) = x^2 - 8x + 19$. Confirm that this is true. Then, find a quadratic function for which $g(2) = -1$ and $g'(2) = 4$.

P13.3 Use implicit differentiation to find the slope of $y^2 = x^3 - x + 1$ at $(-1, -1)$. In other words, find $\dfrac{dy}{dx}$ without first solving for y as an explicit function of x.

P13.4 Given that $P'(t) = k \cdot P(t)$ and $P(0) = 2718$, find $P(t)$ in terms of k.

P13.5 Find $\dfrac{dy}{dx}$ for each of the following curves:

(a) $y = x \cdot 2^x$

(b) $y = \sin\left(\dfrac{1}{x}\right)$

(c) $y = \dfrac{2x}{x^2 - 1}$

(d) $y = \dfrac{e^x - e^{-x}}{2}$

P13.6 For the function $y = \ln(1 - x)$, explain why y is virtually the same as $-x$ when x is close to zero. You might approach this by using a linear approximation for the given function and then compare it to $-x$.

P13.7 Give an example of a function for each of the following scenarios. Compare the graphs of the two functions.

(a) The instantaneous growth rate is a constant 12%.

(b) The annual growth rate is a constant 12%.

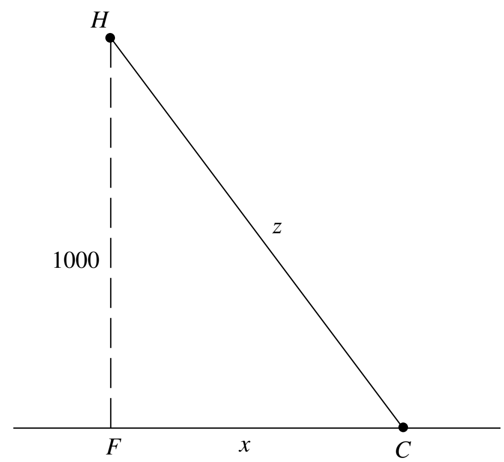
P13.8 Solve the following antiderivative questions. In other words, find F, g, and S.

(a) $F'(x) = 6x^5$

(b) $g'(t) = 10\cos(5t)$

(c) $\dfrac{dS}{du} = \dfrac{1}{2}e^{u} + \dfrac{1}{2}e^{-u}$

P13.9 A police helicopter is hovering 1000 feet above a highway, using radar to check the speed of a car C below. The radar shows that the distance HC is 1250 feet and increasing at 66 ft/sec. Is the car exceeding the speed limit of 65 mph? To figure out the speed of the car, let $x = FC$, where F is the point on the highway that is directly beneath H, and let $z = HC$. Notice that x and z are both functions t and that $z(t)^2 = 1000^2 + x(t)^2$. Differentiate this equation with respect to t. The new equation involves $\dfrac{dx}{dt}$ and $\dfrac{dz}{dt}$, as well as x and z. The information from the radar means that $\dfrac{dz}{dt} = 66$ when $z = 1250$. Use this data to calculate the value $\dfrac{dx}{dt}$.



P13.10 Find a function (this may be an equation, a graph, or a description in words) whose graph $y = f(x)$:

(a) has negative slope, which increases as x increases;

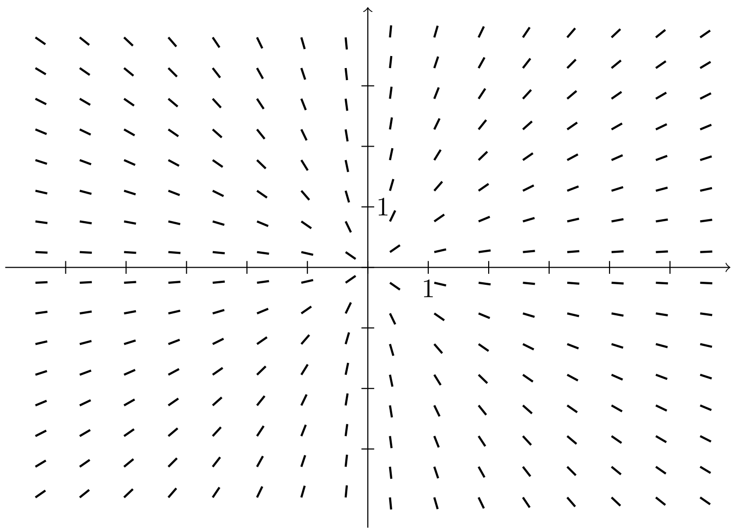
(b) has positive slope, which decreases as x increases.

P13.11 If $0 < f''(a)$, then the graph of $y = f(x)$ is said to be concave up (or have *positive curvature*) at the point $(a, f(a))$. Explain this terminology based on what is happening with the first derivative at $(a, f(a))$. If $f''(a) < 0$, then the graph of $y = f(x)$ is said to be concave down (or have *negative curvature*) at the point $(a, f(a))$. Explain this terminology as well.

P13.12 Examine the following slopefield for a certain differential equation.

(a) Sketch solution curves through the following points: $(1, 1)$, $(-4, 2)$, $(3, -5)$.

(b) What do all the solutions of this differential equation seem to have in common?



P13.13 Consider the function $f(x) = \begin{cases} 4 - 2x, & \text{for} \quad x < 1 \\ ax^2 + bx, & \text{for} \quad x \geq 1 \end{cases}$.

What values of a and b guarantee that f is both continuous and differentiable? To answer this question, it may be helpful to visualize these two functions using a graphing tool and create sliders for a and b. Then, solve algebraically.

P13.14 Find the first and second derivatives of each of the following functions.

(a) $f(z) = z\ln(z)$ **(b)** $f(t) = e^{-t^2}$

P13.15 A function g is continuous on the interval $[-2, 4]$, with $g(-2) = 5$ and $g(4) = 2$. Its derivatives have properties summarized in the table.

x	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
$g'(x)$	positive	undefined	negative	0	negative
$g''(x)$	positive	undefined	positive	0	negative

(a) Find the x-coordinates of all globally extreme points for g. Justify your answer.

(b) Find the x-coordinates of all inflection points for g. Justify your answer.

(c) Make a sketch of $y = g(x)$ that is consistent with the given information.

P13.16 The volume of a cube is increasing at 120 cc per minute at the instant when its edges are 8 cm long. At what rate are the edge lengths increasing at that instant? Consider differentiating both sides of the equation $V = x^3$, with respect to t, to get $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$, and then use the given values.

P13.17 A model for the net result of the attractive and repulsive forces that exist between two atoms is given by $F(x) = \dfrac{c}{x^2} + \dfrac{d}{x^3}$ where x is the separation distance between the atoms, and c and d are constants.

- **(a)** Determine values for c and d which give a maximum attraction of 1 at $x = 1$.
- **(b)** Graph the force function.
- **(c)** Determine where the graph has an inflection point. How is the force changing at the inflection point?

P13.18 A rumor spreads among the students at PEA according to the differential equation $\dfrac{dR}{dt} = 0.08R(1 - R)$, where R is the proportion of the student population who have heard the rumor.

- **(a)** Obtain a slopefield and a solution with initial value $R(0) = 0.03$. If t is in minutes, how long does it take for 90% of the students to hear the rumor?
- **(b)** Explain the shape of the solution curve and what in the differential equation causes this characteristic shape of the *logistic curve*.
- **(c)** Restricting R to the range of 0 to 1, how many different types of solution curves can you perceive by examining the slopefield?
- **(d)** Allowing R to go outside the range of 0 to 1, what other families of solution curves can you perceive by examining the slopefield?

Prelaboratory 13: Introducing Euler's Method

PL13.1 The velocity of an object dropped from the top of a tall building is $y' = -9.8t$, where the height y is in meters, the velocity y' is in meters/second, and time t is in seconds. The height of the building is 130 m and the initial velocity is -15 m/sec.

(a) Using the velocity y', what is the linear approximation for height y centered at the starting point $(0, 130)$?

(b) What is the value of the linear approximation for height after one second?

PL13.2 A population is changing at a rate given by the differential equation $dP/dt = 0.025P$. The initial population at time $t = 0$ is $P_0 = 3600$, where t is in years.

(a) What is the linear approximation for population P centered at the starting point $(0, 3600)$?

(b) What is the value of the linear approximation for population after one year?

Given the derivative $\frac{dy}{dx}$, we have had a lot of practice finding the linear approximation for y. We have noticed, however, that the linear approximation is generally less accurate as you increase the distance from the center of the approximation. One method for achieving a more accurate approximation is a recursive technique known as Euler's method.

Laboratory 13: Introducing Euler's Method

L13.1 The velocity of an object falling from the top of a tall building is $y' = -32t$. The height of the building is 400 ft and the initial velocity is 0 ft/sec. We thus have the initial point $(t_0, y_0) = (0, 400)$. The linear approximation centered at this point is $y = y'(0) \cdot (t - 0) + 400$, where $y'(0)$ is the derivative at the point $(0, 400)$ and thus the slope of the linear approximation at that point.

(a) Confirm that the value of the linear approximation one second later, where $t = 1$, is $y = 400$.

(b) If we want to approximate the function at $t = 2$, we could substitute $t = 2$ into the linear approximation formula. Instead, however, we can use a two-step approximation method by making a new line through the point $(1, 400)$ that is parallel to the tangent to the curve at the point with $t = 1$. This new line has equation $y = y'(1) \cdot (t - 1) + 400$. Confirm that the approximation from this line at $t = 2$ is $y = 368$.

(c) Continue the process of recursively taking the result after each step and substituting this point back into the linear approximation formula to find Euler's method approximation using technology such as a spreadsheet. Fill in the remaining y-values in the table below.

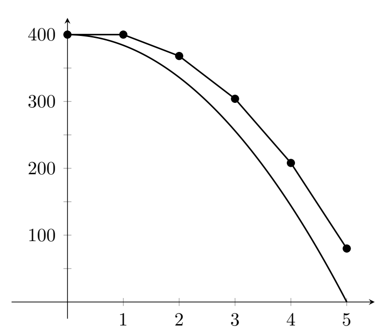
t	0	1	2	3	4	5
y	400	400	368			

L13.2 How do the approximation values in the table compare to the actual y-values? We can find out by solving the differential equation $y' = -32t$ for the curve that contains the point $(0, 400)$.

(a) Explain why the general solution is $y = -16t^2 + C$, and the particular solution is $y = -16t^2 + 400$.

(b) Compare the approximation with the actual value at $t = 5$. Explain how all this relates to the graph below.

(c) In the graph shown, the stepsize for each line segment was $\Delta t = 1$. How would the graph above look if you changed the stepsize to $\Delta t = 0.5$? Use $\Delta t = 0.5$ to find the Euler's method approximation using the spreadsheet option in Geogebra, and examine the accuracy of the approximation after 10 steps, at $t = 5$.



L13.3 A population is changing at a rate given by the differential equation $\frac{dP}{dt} = 0.05P$. The initial population at time $t = 0$ is $P_0 = 3140$, where t is in years. The linear approximation at $(0, 3140)$, which is the first step in Euler's method, is:

$$P_1 = P_0 + 0.05P_0 * \Delta t$$

With $P_0 = 3140$ and a step size $\Delta t = 0.5$, this equation yields $P_1 = 3218.5$.

(a) The next step in Euler's method is to recompute the linear approximation using P_1, which gives $P_2 = P_1 + 0.05P_1 * \Delta t$, the approximation for the population after 1 year. Compute P_2 and explain each term in the equation for P_2. Why is this an approximation for time t = 1 year?

(b) Make a template spreadsheet for Euler's method with columns for t, P, slope (the value of the derivative), and new P (which is P times slope added to P). The new P-value in a row then moves to the P-value of the next row. Fill in the table with values from $t = 0$ up to $t = 10$ using a step size $\Delta t = 0.5$.

(c) Solve the differential equation $\frac{dP}{dt} = 0.05P$ for the particular function $P(t)$ with $P_0 = 3140$. Compare the Euler's method approximation with the actual value of P at $t = 10$.Graph the sequence of points for the Euler's method approximation on the same axes as the exact solution $P(t)$.

(d) Recompute the Euler's method approximation for the population at $t = 10$ using a stepsize of $\Delta t = 0.1$. Graph on the same set of axes the function $P(t)$ and the sequence of points generated by Euler's method. Discuss the graph.

L13.4 A population is constrained by an upper limit 100 and is changing at the rate $\frac{dP}{dt} = 0.08(100 - P)$. The initial population at $t = 0$ is $P_0 = 8$.

(a) Use Euler's Method to generate values for the population from $t = 0$ to $t = 50$ using a stepsize $\Delta t = 1$.

(b) Graph the ordered pairs (t, P) generated with Euler's Method. Is this what you expected for the shape of the graph? What do you expect to see if you continue iterating Euler's method past $t = 50$? How does your graph compare with a graph of the slope field for this differential equation?

(c) The differential equation given in this part is of a type that we have not yet solved. How do we know that the Euler's method approximation gives a reasonably accurate picture of the solution? Is our choice of $\Delta t = 1$ a small enough step size?

L13.5 Write a report that includes an explanation of how Euler's method works, the general recursive equations for Euler's method, and how it can be used to generate a sequence of points that approximate the solution to a differential equation. Include in your report a graph to accompany your explanation. Some questions to consider: Why would we need to use Euler's method? What do the points from Euler's method represent? How is Euler's method related to linear approximation? How is Euler's method affected by step size? How does all of this relate to slopefields? Why do you think Euler's method is also called the tangent line method?

P14.1 Write (but do not try to solve) differential equations given the following descriptions of change.

- **(a)** The temperature T of a cup of water placed in a freezer set to 20 degrees decreases at a rate that is proportional to the difference between T and 20.
- **(b)** The velocity of a car going downhill increases linearly as a function of time.
- **(c)** The rate at which the thickness y of ice increases is inversely proportional to the thickness of the ice.
- **(d)** A population grows at a continuous annual rate of 2%, and immigrants arrive continuously at a rate of 150 per year.

P14.2 At noon, a blue sports car was 15 miles south of an intersection, heading due north along a straight highway at 40 mph. Also at noon, a red sports car was 20 miles west of the same intersection, heading due east along a straight highway at 80 mph.

- **(a)** At noon, when the cars were 25 miles apart, at what rate was their separation decreasing?
- **(b)** At 1 pm, when the cars were 65 miles apart, at what rate was their separation increasing?
- **(c)** At what time of day were the cars closest, and how far apart were they?

P14.3 A population is changing at a rate given by the differential equation $\dfrac{dP}{dt} = 0.12P$. The initial population at $t = 0$ is $P_0 = 144$.

- **(a)** Find the Euler's Method approximation for the population at $t = 6$ using stepsize of $\Delta t = 0.2$.
- **(b)** Solve the differential equation $\dfrac{dP}{dt} = 0.12P$ for the particular function $P(t)$ with $P_0 = 144$. Compare the Euler's Method approximation with the actual value of P at $t = 6$.

P14.4 For each equation, find $\dfrac{dy}{dx}$

- **(a)** $y = \ln(10 - x)$
- **(b)** $\ln y = 3 \ln x$
- **(c)** $2\sqrt{y} = 8 - 0.4x$

P14.5 *Separation of variables.* The circle $x^2 + y^2 = 25$ is a solution curve for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

(a) Confirm that this is true by differentiating $x^2 + y^2 = 25$.

(b) You can solve the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$ by putting all the x's on one side of the equation and all the y's on the other side. The resulting equation $y\dfrac{dy}{dx} = -x$ can be solved by guessing an *antiderivative* function for each side of the equation. Think about the Chain Rule in reverse to find a function with derivative $y\dfrac{dy}{dx}$. What function has a derivative $-x$?

(c) Explain why the solution curves $x^2 + y^2 = C$, where C is a constant, are all solutions for the differential equation in this problem. Verify that the slope field reveals the solution curves.

P14.6 (Continuation) Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$.

(a) Use a slope field to make a conjecture about the solutions to this differential equation.

(b) Solve the differential equation by the method of separation of variables. In other words, multiply both sides by y and then find the antiderivative functions for each side of the equation. To find the general solution, you will want to include a constant in each antiderivative. What can you do when the two constants are added or subtracted?

(c) Give a general equation for the solution curves for this differential equation.

P14.7 *Constrained population growth.* A population P (in millions) grows in proportion to how much more it has left to grow before it reaches its upper limit L.

(a) For this population, complete the differential equation: $\dfrac{dP}{dt} = k(\qquad\quad)$.

(b) Let $k = 0.05$ and $L = 100$ (million). Obtain a slopefield for this differential equation and confirm that the solution curves level off at 100.

P14.8 A population is constrained by an upper limit 100 and is changing at the rate $\dfrac{dP}{dt} = 0.04(100 - P)$. The initial population at $t = 0$ is $P_0 = 8$.

(a) Use Euler's Method to generate values for the population from $t = 0$ to $t = 100$ using a stepsize $\Delta t = 1$.

(b) Graph the ordered pairs (t, P) generated with Euler's Method. Is this what you expected for the shape of the graph?

(c) For an extra challenge, solve the original differential equation and graph the solution on the same set of axes as the ordered pairs from Euler's Method.

P14.9 You have solved some separable differential equations by writing them in the form $f(y)\dfrac{dy}{dx} = g(x)$. Show that $\dfrac{dy}{dx} = 2xy^2$ is also of this type by writing the equation in this form. Find the solution curve that goes through the point $(5, 1)$. Confirm your answer with Geogebra using slopefields and SolveODE.

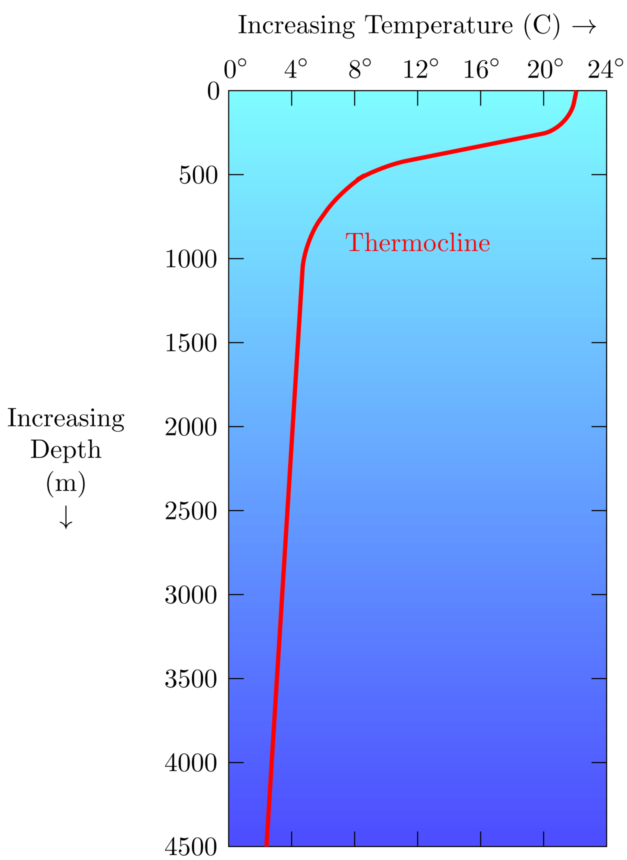
P14.10 Boyle's Law states that for a gas at constant temperature, the pressure times the volume of the gas is constant, or $PV = c$.

(a) Turn this into a rate problem by taking derivatives with respect to time. In other words, apply the operator $\dfrac{d}{dt}$ to both sides of the equation.

(b) If the volume is decreasing at a rate of 10 cm^3/sec, how fast is the pressure increasing when the pressure is 100 N/cm^2 and the volume is 20 cm^3?

P14.11 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x}$ by separation of variables. You will need to divide both sides by y. Find the particular solution that contains $(4, 2)$.

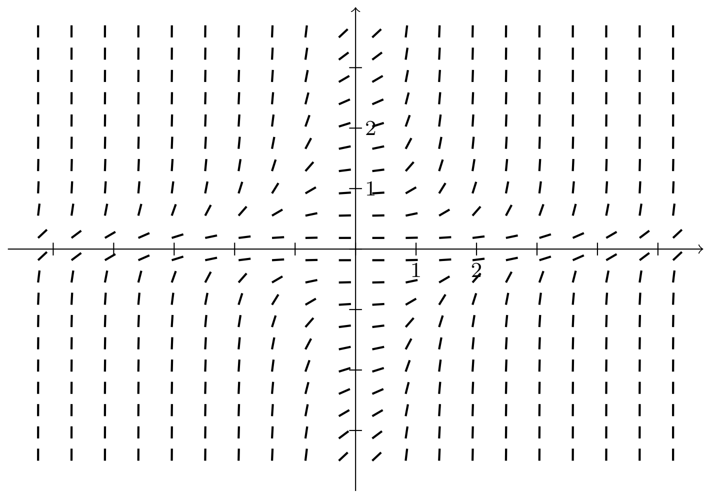
P14.12 The graphic displays the temperature of the ocean water versus depth. Describe, in words and with a graph, the rate of change of the temperature as a function of depth.



P14.13 Examine the slopefield shown below.

(a) Sketch solution curves through the following points: $(-4,-1)$, $(2,2)$, $(4,-2)$.

(b) What general behaviors of the solution curves can you deduce from the slopefield?



Prelaboratory 14: Extreme Skydiving

PL14.1 We can simplify the differential equations for a falling object (such as a skydiver) by ignoring air resistance, which leads to an equation with constant acceleration due to gravity represented by:

$$\frac{dv}{dt} = -g$$

The term dv/dt represents acceleration, and g is the acceleration of gravity equal to 9.8 m/sec^2. (The upward direction is positive and the downward direction is negative; hence g has a negative sign.) Solve this equation for the velocity as a function of time given an initial velocity of v_0.

PL14.2 The equation for $v(t)$ found in the previous question is also the equation for the derivative of height h. Use $v(t)$ in the equation below and solve for $h(t)$ given an initial height h_0:

$$\frac{dh}{dt} = v(t)$$

PL14.3 Unlike the situation with projectile motion, we must include air resistance to have an effective model for skydiving (otherwise what's a parachute for?). As a skydiver descends, the force of air resistance is in the upward direction, since it opposes the direction of motion. Air resistance increases as speed increases (up to a point), and we will assume it is proportional to the velocity squared. The differential equation for acceleration is therefore:

$$\frac{dv}{dt} = -g + kv^2$$

The constant of proportionality is

$$k = \frac{C\sigma A}{2m}$$

where $C = 0.57$ is the coefficient of air resistance; $\sigma = 1.225$ kg/m^3 is the density of air; $A = 0.7$ m^2 is the surface area of the skydiver before the parachute opens; and $m = 75$ kg is the mass of the skydiver.

(a) Calculate the value of k.

(b) With an initial velocity of 0 m/s, use Euler's method to calculate approximate values for the velocity for the first 2 seconds of freefall using a step size of 0.5 seconds.

PL14.4 To prepare for this investigation, first watch the video of Extreme Skydiving. Pay close attention to the shapes of the velocity, force, and altitude curves shown in the video. We will be attempting to obtain similar results in our work.

Laboratory 14: Extreme Skydiving, Part 1

L14.1 To model the descent of a parachutist, the equation for acceleration is

$$\frac{dv}{dt} = -g + kv^2$$

The constant of proportionality is $k = \frac{C\sigma A}{2m}$, with parameter values as given in the prelab.

L14.2 In contrast to the equations of motion without air resistance, the equation with air resistance is one that we cannot solve symbolically with an anti-derivative. Use Euler's method with a step size of $\Delta t = 0.5$ to generate approximate values for the velocity during the first 20 seconds of free fall. State the units of velocity.

L14.3 Obtain a graph of velocity vs. time and discuss the important features of the graph. What is the terminal velocity of the skydiver? Solve for this from the differential equation for dv/dt and compare with your data and graph.

L14.4 Suppose the parachute is opened after reaching terminal velocity in free fall. Assume the skydiver is using an old-fashioned military parachute with a surface area of 30 m^2. What is the new value of k given the change to $A = 30$? What is the terminal velocity of the parachute? This will be the landing speed for the skydiver. Is it reasonably safe?

L14.5 Use Euler‘s method to simulate free fall and then opening the parachute at 20 seconds. Obtain one set of data for the first 30 seconds and show a graph of the velocity versus time. Is your solution what you expected? What happened to your calculations when you opened the parachute? What modifications to the step size $\Delta t = 0.5$ did you make to get a reasonable solution? Why? Give a visualization of what was happening graphically that led you to modify the step size.

L14.6 You can now use the sequence of velocities to approximate the height of the skydiver with the recursive equation

$$h_n = h_{n-1} + v_{n-1} * \Delta t.$$

Because $dh/dt = v$, the previous equation is an example of *numerical integration.*

L14.7 Generate a graph of the height vs. time of a skydiver jumping out of a plane at a height of 1000 m and opening the parachute after 20 seconds. Extend the calculations until the skydiver lands on the ground. What are the important features of the height vs. time graph?

L14.8 Write a paragraph that explains your understanding of how you utilized Euler's method to "solve" this problem, the important features (such as linear pieces, approaching a constant, any discontinuity, etc.) of the graphs you obtained (in context), how your solution compares to the graphs of the Extreme Skydiving, and any challenges you faced during this investigation.

Laboratory 14: Extreme Skydiving, Part 2

L14.9 Our Part 1 results are consistent with the latter half of the Extreme Skydiving graphs from the video. Why did the first part of the velocity and height graphs not match our model?

L14.10 A relatively uncomplicated function for air density is $1.225(0.903)^h$, where h is the height above sea level in kilometers. Your challenge is to incorporate variable air density into your model and replicate the Extreme Skydiving results, opening the parachute at some safe height above the eventual landing spot. This will require some careful use of the standard spreadsheet fill-down capability. Since air density depends on height, and velocity depends on air density, you will need to fill down all three values simultaneously. You can't compute all the velocity values, then go back and compute all the heights, as we did in Part 1. How do your velocity and height graphs compare to the graphs from the Extreme Skydiving video?

L14.11 Write up an explanation of your work to go with your data and graphs for this part, along with an evaluation of your results.

Mathematics 43C

Mathematics Department
Phillips Exeter Academy
Exeter, NH
August 2024

P15.1 Solve the following antiderivative questions and thus find F, h, and A.

(a) $dF = \frac{1}{u^2}du$

(b) $h'(t) = \sin(2t)$.

(c) $dA = \frac{1}{2}e^x - \frac{1}{2}e^{-x}dx$

P15.2 Find dy/dx for the function defined implicitly by $y = xy^2 + 1$.

P15.3 Find at least two different functions S for which $S'(x) = x^2$.

P15.4 Graph $y = \cos(x)$ and find and graph $\frac{dy}{dx}$ How does the graph of the derivative explain the behavior of the graph of y?

P15.5 Find antiderivatives for the following:

(a) $\frac{1}{y}dy$

(b) $(\sin(x))^2 \cos(x)$

(c) $e^u * u'$

P15.6 Find the derivatives of the reciprocal functions of cos, sin, and tan, namely the sec (secant), csc (cosecant), and cot (cotangent) functions.

P15.7 Find the derivatives of the following: x^π, π^x, π^π?

P15.8 Find functions with the following derivatives:

(a) $10x^9$

(b) x^{99}

(c) $\frac{1}{2\sqrt{x}}$

(d) $\sqrt{x}$

P15.9 (Continuation) Given that n is a constant, find the antiderivatives of nx^{n-1} and x^n.

P15.10 Graph $y = \sin(x)$ and find and graph $\frac{dy}{dx}$. How does the graph of the derivative explain the behavior of the graph of y?

Prelaboratory 15: The Fundamental Theorem of Calculus

PL15.1 The *arithmetic series* $3 + 7 + 11 + \ldots + 159$ is a sum of 40 terms that increase by the constant difference 4.

(a) Confirm that each term is 3 plus a multiple of 4. For example, the third term is $3 + 4 * 2$, and the fifth term 19 (not shown) is $3 + 4 * 4$. Without calculating all the terms before it in the series, what is the 10th term? What is a formula for the nth term in terms of n?

(b) The Greek letter sigma can be used as a shorthand way of writing out the sum of this series as follows:

$$\sum_{n=1}^{40}(3 + 4 * (n - 1))$$

The letter n acts as an index that counts the terms from 1 to 40. Each term is defined by $(3 + 4 * (n - 1))$, which is the nth term that we were looking for in part (a). Evaluate this summation expression using a calculator that has sigma notation available.

PL15.2 Find the following sums using a calculator with sigma notation:

(a) $\displaystyle\sum_{n=0}^{9} 12 * (3/5)^n$

(b) $\displaystyle\sum_{n=4}^{12} 5n$

(c) $1 + 9 + 17 + \ldots + 1001$ [Hint: there are 126 terms]

(d) $1 + 4 + 9 + \ldots + 361$ [Hint: $361 = 19^2$]

PL15.3 The solution to the differential equation $dy/dx = 2x + 1$ can be approximated with Euler's method starting from the point $(0, 0)$ using a stepsize of $\Delta x = 1$ and ending at $x = 10$.

(a) Starting at the initial value $y_0 = 0$, confirm that the approximation for $x = 1$ is $y_1 = 1$.

(b) Show that the next approximation is $y_2 = 4$, and the third y-value is $y_3 = 9$.

(c) Find the approximation at $x = 10$ by calculating the value of the summation below, and explain each part of the summation expression.

$$\sum_{n=0}^{9}(2n + 1)$$

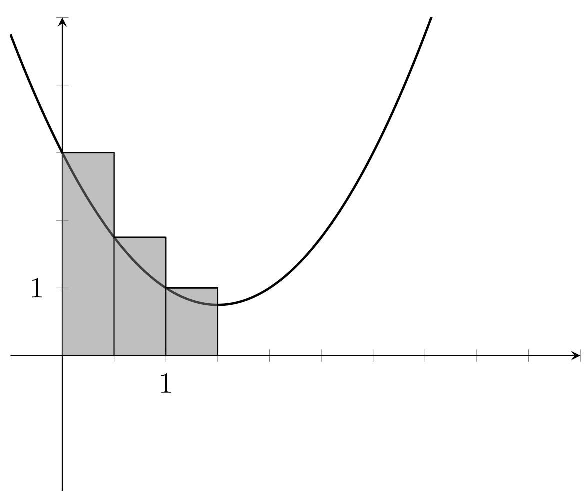
Laboratory 15: The Fundamental Theorem of Calculus

L15.1 Use Euler's method to approximate the function $y = f(x)$ that has derivative $\frac{dy}{dx} = x^2 - 3x + 3$ and initial value $(0, 1)$. Use a stepsize of $\Delta x = 0.5$, and generate a sequence of values from $x_0 = 0$ to $x_n = 4$. Fill in the table of values below.

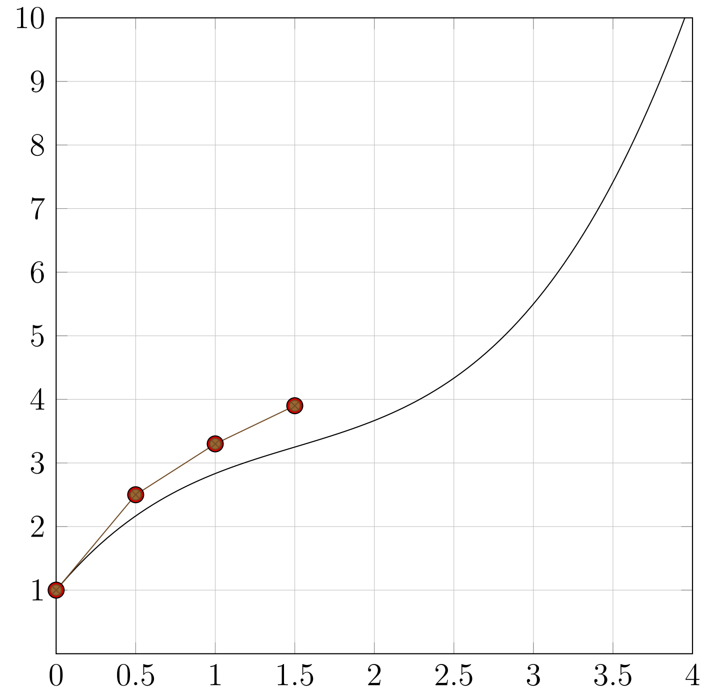
x	y	Slope or $\frac{dy}{dx}$	New y
0	1	3	2.5
0.5	2.5	1.75	3.375
4	9.5	7	13

L15.2 Use Geogebra to generate the slopefied for the given differential equation. Plot the points you found in the table above. Do they appear to be good approximations given the slope field?

L15.3 A graph of $\frac{dy}{dx}$ is shown below. A sequence of rectangles with a base on the x-axis, a width of Δx, and a corner on the $\frac{dy}{dx}$ curve is shown. Draw the additional rectangles up to where $x = 4$. Calculate the areas of a few of the rectangles you've drawn. Which column in the table above gives the heights of these rectangles? What product gives the area of each rectangle?



L15.4 On the set of axes below, plot the points from #L15.1. The first few points are shown. The result is an approximation for the graph of the solution of the differential equation $dy/dx = x^2 - 3x + 3$. What quantities in the previous graph correspond to the Δy values below? What is the equation for the solution curve shown? How can you adjust Euler's method to make the approximation more accurate?



L15.5 The initial y-value is 1, which we symbolize by $y_0 = 1$. The first y-value y_1 generated by Euler's method is found by adding the slope times Δx (which is 0.5) to y_0. This can be symbolized by

$$y_1 = y_0 + \left.\frac{dy}{dx}\right|_0 \cdot \Delta x.$$

Describe in words how to generate the second y-value, then fill in the blank below:

$$y_2 = y_1 + \underline{\hspace{3cm}}$$

L15.6 Show by substitution that y_2 can be written as:

$$y_2 = y_0 + \left.\frac{dy}{dx}\right|_0 \cdot \Delta x + \left.\frac{dy}{dx}\right|_1 \cdot \Delta x.$$

How does this question relate to the graph shown in L15.3? How does it relate to the graph shown in L15.4?

L15.7 Fill in the next term in the equation for y_3 below.

$$y_3 = y_0 + \left.\frac{dy}{dx}\right|_0 \cdot \Delta x + \left.\frac{dy}{dx}\right|_1 \cdot \Delta x + \underline{\hspace{3cm}}$$

On the graph in #L15.3, draw the next rectangle that corresponds to the term in the blank. On the graph in #L15.4, draw the next line segment that corresponds to the term in the blank.

L15.8 Continue the substitution pattern in the sequence of equations below.

$$y_4 = y_0 + \left.\frac{dy}{dx}\right|_0 \cdot \Delta x + \left.\frac{dy}{dx}\right|_1 \cdot \Delta x + \underline{\hspace{3cm}} + \underline{\hspace{3cm}}$$

$$y_5 = y_0 + \underline{\hspace{3cm}}$$

$$\vdots$$

$$y_n = y_0 + \left.\frac{dy}{dx}\right|_0 \cdot \Delta x + \left.\frac{dy}{dx}\right|_1 \cdot \Delta x + \cdots + \left.\frac{dy}{dx}\right|_{n-1} \cdot \Delta x$$

L15.9 Relate the rectangles in the graph of #L15.3 to the terms in the last equation above.

L15.10 Draw line segments in the graph of #L15.4 that show the Euler's method approximation of all the terms in L15.8.

L15.11 Use summation notation to show that

$$y_n = y_0 + \sum_{k=0}^{n-1} \left.\frac{dy}{dx}\right|_k \cdot \Delta x$$

and thus

$$y_n - y_0 = \sum_{k=0}^{n-1} \left.\frac{dy}{dx}\right|_k \cdot \Delta x$$

L15.12 Explain how the summation expression for $y_n - y_0$ can be interpreted as an approximation for the area under a curve and as an approximation for the net change in Euler's method. Refer to the graphs of #L15.3 and #L15.4.

L15.13 To improve the Euler's method approximation above, we should _______________ the step size Δx. As the step size Δx _______________, the number of steps n _______________. What are the resulting limits of the approximations of L15.11?

L15.14 We can generalize the previous work by letting the initial value be at the point with $x_0 = a$ and the last value be at $x_n = b$. Starting with the differential equation $dy/dx = f(x)$, we call the solution function $F(x)$, which is an antiderivative of $f(x)$. This implies $y_0 = F(a)$ and y_n approximately equal $F(b)$. Explain how this leads to the following equation:

$$F(b) - F(a) = \lim_{\Delta x \to 0} \sum_{k=0}^{n-1} \left. \frac{dy}{dx} \right|_k \cdot \Delta x$$

The expression on the right side of the equation above defines the definite integral from a to b where $f(x)$ has been substituted for dy/dx. What are $f(x)$ and $F(x)$ in the example from this lab?

L15.15 We can generalize the previous work by letting the initial value be at the point with $x_0 = a$ and the last value be at $x_n = b$. Complete the expressions for the first two y-value approximations:

$$y_1 = y_0 + \left. \frac{dy}{dx} \right|_0 \cdot \Delta x \qquad \text{and} \qquad y_2 = y_1 + \underline{\hspace{3cm}}.$$

L15.16 *The Fundamental Theorem of Calculus.* The two equations in L15.14 can be combined into one equation

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

(a) Evaluate the integral

$$\int_0^4 (x^2 - 3x + 3)\,dx$$

(b) Give two interpretations of this value consistent with the graphs of #L15.3 and #L15.4.

(c) The value of this integral equals the _______________ of the antiderivative of _______________ from x = _______________ to x = _______________ . AND The value of this integral equals the _______________ under the curve _______________ from x = _______________ to x = _______________ .

L15.17 How does the Fundamental Theorem of Calculus (FTC) link derivatives and antiderivatives? The FTC links the two halves of calculus *Differential Calculus* and *Integral Calculus*.

L15.18 Compile a report with answers and graphs for all the items in this lab.

P16.1 Find the values of the following integrals by using antiderivatives

(a) $\displaystyle\int_0^3 \left(x^2+2x+1\right) dx$ **(b)** $\displaystyle\int_{-1}^1 e^x\, dx$ **(c)** $\displaystyle\int_1^{10} \frac{1}{x}\, dx$

P16.2 Some of the particular solutions of the differential equation $\dfrac{dL}{dt} = k(M-L)$ are called *learning curves*, where L is the amount of learning of a body of facts or of a particular skill.

(a) Finish the following statement: The rate of learning is proportional to ...

(b) In a learning curve, the initial value of L is less than M. What is the shape of the curve? What is the significance of the parameter M?

P16.3 Determine antiderivatives of the following. Check your work using differentiation.

(a) $\dfrac{1}{3x}$ **(b)** $\dfrac{1}{5+x}$ **(c)** $\dfrac{1}{10-0.2x}$

P16.4 The differential equation $\dfrac{dP}{dt} = kP\left(1-\dfrac{P}{M}\right)\left(\dfrac{P}{m}-1\right)$ is called the *threshold model* for population growth. The parameter k is a constant of proportionality, M is the maximum attainable population, and m is called the threshold value of the population ($m < M$). Refer to this differential equation to answer the following questions.

(a) Use technology to obtain a slope field for this differential equation with parameter values $k = .05$, $m = 30$, $M = 100$.

(b) Why does the population have to be greater than m to grow? What happens to the population if $P < m$? Why does the population decline if the value of P exceeds M?

(c) Sketch a typical solution curve given a starting population $P(0)$ between m and M.

P16.5 To solve the following separable differential equation for the particular solution with initial value $y_0 = 1000$, it is tempting to add $0.5y$ to both sides of the equation, but the left side will not be easy to antidifferentiate. Instead, if you divide both sides by $(20-0.5y)$, then the left side will be a derivative of an *ln* function.

$$\frac{dy}{dx} = 20 - 0.5y$$

P16.6 How can you use calculus to determine the distance traveled by a vehicle over an interval of time if you are given an explicit function $f(t)$ for the speed of the vehicle at any time t?

P16.7 Find antiderivatives of the following:

(a) $\sqrt{y(t)}\, y'(t)$

(b) $f'(t)\sin f(t)$

P16.8 Not wanting to be caught exceeding the speed limit, the driver of a red sports car suddenly decides to slow down a bit. The table below shows how the speed of the car (in feet per second) changes second by second. Estimate the distance traveled during this 6-second interval.

Time	0	1	2	3	4	5	6
Speed	110.0	99.8	90.9	83.2	76.4	70.4	65.1

P16.9 (Continuation) The red sports car's speed during the time interval $0 \le t \le 6$ is actually described by the function $f(t) = \dfrac{44000}{(t+20)^2}$. Use this function to calculate the exact distance traveled by the sports car.

P16.10 Find the antiderivative of

(a) e^{3x} **(b)** $e^{-\frac{x}{10}}$ **(c)** Ae^{kx}, where A and k are constants

P16.11 The speed in ft/sec of a red sports car is given by $f(t) = \dfrac{44000}{(t+20)^2}$ for $0 \le t \le 6$. It may be helpful to use this Desmos visualization while completing this problem.

(a) Explain why the sums $\displaystyle\sum_{k=0}^{5} 1 \cdot f(k)$ and $\displaystyle\sum_{k=1}^{6} 1 \cdot f(k)$ are reasonable approximations to the distance traveled by the car during this 6-second interval. Why is the true distance between these two estimates?

(b) Explain why $\displaystyle\sum_{k=0}^{11} 0.5 \cdot f(0.5k)$ is a better approximation than $\displaystyle\sum_{k=0}^{5} 1 \cdot f(k)$.

(c) Explain why $\displaystyle\sum_{k=0}^{119} 0.05 \cdot f(0.05k)$ is an even better approximation.

(d) How is the sum in part (c) related to the Euler's method of approximation?

P16.12 (Continuation) The speed of a red sports car is given by $f(t) = \dfrac{44000}{(t+20)^2}$ for $0 \le t \le 6$.

(a) Given a large positive integer n, the sum $\displaystyle\sum_{k=1}^{n} \frac{6}{n} \cdot f\left(\frac{6k}{n}\right)$ is a reasonable estimate of the distance traveled by the sports car during this 6-second interval. Explain why.

(b) Another reasonable estimate is $\displaystyle\sum_{k=0}^{n-1} \frac{6}{n} \cdot f\left(\frac{6k}{n}\right)$. Compare it to the preceding.

(c) What is the significance of the expression $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{6}{n} \cdot f\left(\frac{6k}{n}\right)$? What is the value of this expression? (You have found this value in an earlier exercise.)

P16.13 Let $f(x) = \cos(\ln x)$ and $g(x) = \ln(\cos x)$.

(a) Describe the domain and range of f and g. Graphs may be helpful here.

(b) Find $f'(x)$ and $g'(x)$.

P16.14 Problems #P16.11 and #P16.12 were about accumulating velocity to calculate the distance traveled by a sports car. The distance is approximated to any degree of accuracy by a *sum of products*, specifically rate times time. The precise answer is a *limit* of such sums. For $0 \le t \le 6$ and $\Delta t = 6/n$, the distance traveled is $\lim_{n\to\infty} \sum_{k=1}^{\infty} f(k\Delta t)\Delta t = \int_0^6 f(t)dt$. The expression $\int_0^6 f(t)dt$ for the exact distance traveled is read "the *integral* of $f(t)$ from $t = 0$ to $t = 6$". The integration symbol is an elongated S - the initial letter of *sum*. The symbol dt, although it does not stand for a specific value, represents the Δt in the approximating sum, and it identifies the *integration variable*.

(a) Since $f(t)$ is velocity, the distance traveled, namely $\int_0^6 f(t)dt$, is the antiderivative of $f(t)$. Thus, the accumulation of velocities (which is an Euler's method problem) can be calculated by using an antiderivative. Find the antiderivative of $f(t)$, which we will call $F(t)$, then take the difference $F(6)-F(0)$ to find the distance traveled.

(b) Explain how this accumulation problem is also an *area* problem, specifically the area under the curve $y = f(t)$ between $t = 0$ and $t = 6$.

(c) Explain how this one problem is simultaneously a problem about accumulating change over an interval to find the net change in one function *and* a problem about accumulating area to find the area under a curve defined by another function.

P16.15 (Continuation) Suppose the speed of a car in ft/sec is given by $f(t) = 90e^{-0.1t}$.

(a) Write an integral for the distance traveled from time $t = 0$ to $t = 5$ seconds.

(b) Write an integral for the distance traveled from time $t = 5$ to $t = 10$ seconds.

(c) Evaluate each of the integrals in (a) and (b).

P16.16 In years past many countries disposed of radioactive waste by sealing it in barrels and dumping it in the ocean. The acceleration of the barrel is the sum of the downward force of gravity, the upward force of the buoyancy of the barrel, and the drag force of the water that is directed opposite to the motion. The forces of gravity and buoyancy are constant, and for a 55 gallon barrel filled with 527 pounds of waste, the acceleration due to gravity minus buoyancy is a constant 3.49 ft/sec^2. The acceleration due to the drag force is proportional to the velocity with a constant of proportionality 0.0049. Thus the differential equation for the velocity of the barrel is

$$\frac{dv}{dt} = 3.49 - 0.0049v.$$

(a) Explain the significance of the terms in this differential equation.

(b) Solve this separable equation to find the velocity as an explicit function of t.

(c) Velocity is the derivative of position, so $v = \dfrac{dh}{dt}$, where h is the height of the object above the ocean floor. Do one more antidifferentiation to find h as a function of t.

(d) A barrel is dumped in an area with an ocean depth of 300 feet. How long will it take for the barrel to reach the ocean floor?

(e) Tests have shown that the barrels tend to rupture if they land with a velocity greater than 40 ft/sec. Will the barrels remain intact when dropped a depth of 300 feet?

Note: Since 1993, ocean dumping of radioactive waste has been banned by international treaty.

P16.17 Sterling, a student of Calculus, was given an assignment to find the derivative with respect to x of five functions. Below are Sterling's five answers. For each, reconstruct the expression that Sterling differentiated, and write your answers in terms of y and x. Can you be sure that your answers agree with the questions on Sterling's assignment?

(a) $2.54 + \dfrac{dy}{dx}$ **(b)** $\dfrac{dy}{dx}\sec^2 y$ **(c)** $\sqrt{5y}\,\dfrac{dy}{dx}$

(d) $\dfrac{7}{y^2}\dfrac{dy}{dx}$ **(e)** $(y - \cos x)\left(\dfrac{dy}{dx} + \sin x\right)$

P16.18 *Displacement* is the difference between the initial position and the final position of an object. It is related to velocity by the process of antidifferentiation. Let $x(t)$ be the position of an object moving along a number line. Suppose that the velocity of the object is $\dfrac{dx}{dt} = 4 - 3\cos 0.5t$ for all t. Calculate and compare

(a) the displacement of the object during the interval $0 \le t \le 2\pi$;

(b) the distance traveled by the object during the interval $0 \le t \le 2\pi$.

Prelaboratory 16: The Gini Index

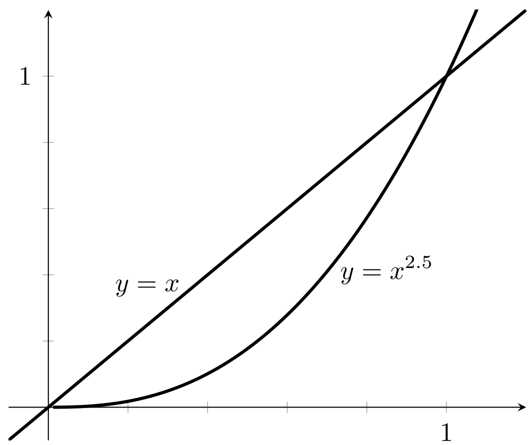
PL16.1 Find the values of the following integrals

(a) $\int_0^1 x dx$ **(b)** $\int_0^1 x^2 dx$ **(c)** $\int_0^1 x^3 dx$

PL16.2 Find the area under the curve $y = f(x)$ between $x = 0$ and $x = 1$ for

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

PL16.3 Evaluate an integral to find the area of the region bounded by $y = x$ and $y = x^{2.5}$.



PL16.4 The process of quantifying inequity in a society's income distribution starts with aggregating the income in each fifth of the population. You will be assigned a society with 25 family units, each with a particular income. Your task is to order the incomes from smallest to largest, then total up the incomes in each fifth (which is five incomes). The lowest fifth will consist of the lowest five incomes, the second fifth will have the next five lowest incomes, and so on. Fill in the chart below with the sum of the incomes in each fifth (in column 2), then convert these values to the percent of total income (in column 3).

Fifth of population	Aggregate income	Percent aggregate income
Lowest fifth		
Second fifth		
Third fifth		
Fourth fifth		
Highest fifth		
Total	2,000,000	100

Table 1: Income Distribution

Laboratory 16: The Gini Index

L16.1 Complete the work in the last prelab question and check to see that you have the correct values in Table 1 for your particular group of 25 incomes.

L16.2 Add the fifths for percent aggregate income to fill in Table 2 with cumulative percent aggregate income. For example, the entry for the Lowest three-fifths is the sum of the lowest fifth, second fifth, and third fifth from Table 1.

Fifth of population	Cumulative percent aggregate income
Lowest one-fifth	
Lowest two-fifths	
Lowest three-fifths	
Lowest four-fifths	
All fifths	100

Table 2: Cumulative Income

L16.3 Plot the data in Table 2 as the ordered pairs with aggregate proportion of population (the numbers $0.0, 0.2, 0.4, 0.6, 0.8, 1.0$) on the horizontal axis and proportion of cumulative percent aggregate income (from Table 2, converted to proportion) on the vertical axis. The accompanying graph is what a plot might look like. A curve that fits these points is called a *Lorenz curve*.

L16.4 Discuss the following questions in your group.

(a) Why must a Lorenz curve be increasing?

(b) Why must a Lorenz curve be concave up (unless it is the line segment from $(0, 0)$ to $(1, 1)$)?

L16.5 Equity/Inequity

(a) Sketch the Lorenz curve for a society with perfect *equity* of income distribution, meaning income is distributed equally throughout the population.

(b) On the same axes, sketch the Lorenz curve for a society with perfect *inequity* of income distribution, which corresponds to one household earning all the income and everyone else earning nothing.

L16.6 Compare your results with the other groups in your class. What do you notice about the Lorenz curve for most equitable income distribution? For the least equitable? And the ones in between? If you sketch all the curves on the same axes, do you notice any ordering related to how equitable is the income distribution?

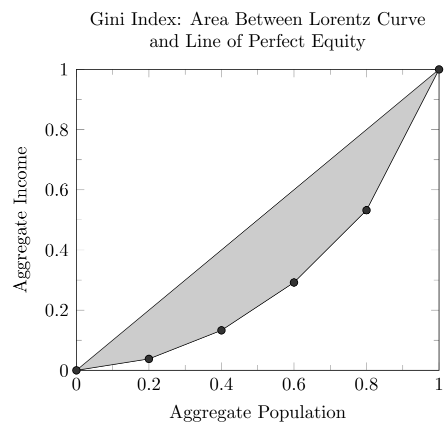
L16.7 A typical Lorenz curve lies between the extremes sketched in the previous exercise. Corrado Gini first defined the Gini index (or Gini coefficient) in 1912, and it has been widely used by economists since. The Gini index is based upon the area of the region bounded by the line of perfect equity and the Lorenz curve, as shown below. To scale the Gini index on the interval from 0 to 1, this area is then divided by the area bounded by the extremes of perfect equity and perfect inequity.

(a) Find the area bounded by the extremes of perfect equity and perfect inequity. Thus, what must the area bounded by the Lorenz curve be multiplied by?

(b) What is the Gini index for a society with perfect equity of income distribution? Perfect inequity of income distribution?

(c) Estimate the Gini index for the income distribution given above.

(d) Estimate the Gini index for the income distribution shown in the graph.



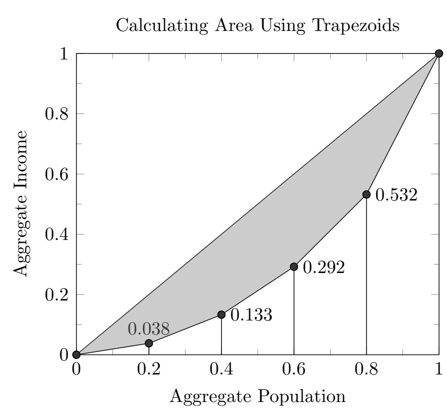
L16.8 We can find the area needed to calculate the Gini index by finding the area below the Lorenz curve (that is, bounded by the Lorenz curve and the x-axis) and subtracting it from the area of the triangle highlighted in the figure above. You can estimate the area under the Lorenz curve without actually finding an equation for the curve. One method is to join the successive data points with line segments and draw trapezoids as shown in the figure below.

(a) Your graph will be somewhat similar to the graph shown. For your graph, label the coordinates of the points on the line graph for aggregate income proportion and draw vertical line segments to form trapezoids. Find the areas of the five trapezoids.

(b) Use the total of the areas of the trapezoids to determine the area of the polygonal region bounded by the line of perfect equity and the line graph of data points.

(c) Calculate the area for your assigned income distribution, and hence, calculate the Gini index for your group.

(d) Compare your results with the Gini indices for the other groups.



L16.9 The relevant area for the Gini index can also be estimated by evaluating an integral. To do so, however, requires that we have an explicit formula $L(x)$ for the Lorenz curve.

(a) Confirm that the area between the line of perfect equity and the Lorenz curve is given by

$$\int_0^1 x\ dx - \int_0^1 L(x)\ dx.$$

The Gini index is therefore 2 times the value of this expression.

(b) A simple expression for $L(x)$ if the power function x^n. Notice that this function contains the points $(0,0)$ and $(1,1)$, and it is also increasing and concave up, so it meets the requirement for a Lorenz curve. Evaluate the expression for the Gini index in part (a), and show that the Gini index can be simplified to $(n-1)/(n+1)$.

(c) Find a value of n that give a good fit of x^n with your graph from problem L16.8. Use your result from part (b) to calculate a value for the Gini index. Compare your answer with your Gini index calculated using trapezoids..

(d) Use integrals to find the area in (a), hence calculate a value for the Gini index.

L16.10 The accompanying Excel spreadsheet contains the U.S. income distribution quintiles for the years from 1967 to 2021.

(a) For the year 2021, enter the necessary formulas in the 2021 row to calculate the Gini index using the trapezoid method. You should get a result of approximately 0.456.

(b) Replicate your formulas for all of the other years in the spreadsheet, which will give you a historical record of Gini indices in the U.S. since 1967.

(c) Plot a graph of Gini index versus year. What do you notice? Is there a trend in your results?

L16.11 Write a lab report with a discussion of the results of the groups of 25 incomes, an explanation of how you calculated the Gini index, a graph of your historical results for the U.S. with a sample calculation for one year of your choosing, and a discussion of the historical trend in the U.S. Gini index.

P17.1 Because of the close connection between integrals and derivatives, integral signs are often used to denote antiderivatives. Thus statements such as $\displaystyle\int x^2\,dx = \frac{1}{3}x^3 + C$ are common. The term *indefinite integral* often appears as a synonym for *antiderivative*, and an integral such as $\displaystyle\int_0^1 x^2\,dx$ is often called *definite*. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \cos 2x\,dx$ **(b)** $\displaystyle\int e^{-x}\,dx$ **(c)** $\displaystyle\int 110(0.83)^x\,dx$

P17.2 *Immigration model for population growth.* The population of Gravesend grows in proportion to its current population with a constant of proportionality 0.03. It also grows due to a constant immigration rate of 100 000 people per year. The current population is 10 million.

(a) Explain why the differential equation $\dfrac{dP}{dt} = 0.03P + 0.1$ can be used to describe the population growth in Gravesend.

(b) Use Euler's method to estimate the population for the next 30 years. How long will it take for the population to double its current size? When will it reach 100 million?

(c) How does your Euler's method estimate relate to the value of the definite integral $\displaystyle\int_0^{30} 0.03P(t) + 0.1\,dt$?

(d) You can actually find a symbolic formula for $P(t)$ by solving the given differential equation. You solved a differential equation similar to this in #P14.8. Solve the differential equation, then compare the exact values from your solution with the estimates in part (b).

P17.3 Apply the Fundamental Theorem of Calculus by using an antiderivative to find the area of the first quadrant region that is enclosed by the coordinate axes and the parabola $y = 9 - x^2$.

P17.4 "Common integration is only the memory of differentiation." (Augustus De Morgan) Explain what you think is the meaning of this quote.

P17.5 Evaluate: **(a)** $\displaystyle\int_0^{\pi} \cos x\,dx$; **(b)** $\displaystyle\int_0^{\pi} \sin x\,dx$. Could you have expected the result based on your knowledge of the graphs? Explain.

P17.6 Being successful when using antidifferentiation to evaluate an integral often depends upon recognizing the *form of the integrand*. To illustrate this remark, the following definite integrals have a common form. Evaluate each of them.

(a) $\displaystyle\int_6^{15} 2x\sqrt{64 + x^2}\,dx$ **(b)** $\displaystyle\int_0^{\pi/2} \cos x\sqrt{\sin x}\,dx$

(c) $\displaystyle\int_0^{1} e^x\sqrt{3 + e^x}\,dx$ **(d)** $\displaystyle\int_1^{e} \frac{1}{x}\sqrt{\ln x}\,dx$

The following three problems involve further exploration of integration as accumulation.

P17.7 The speed in mph of a red sports car as the driver slows to be under the speed limit is described by $s(t) = 80(0.97)^{t^2}$ for $0 \le t \le 5$.

(a) Complete the table below.

t	0	1	2	3	4	5
$s(t)$						

(b) Use a graphing application to construct a speed versus time graph.

(c) Use the points in (a) to estimate the distance traveled during the entire 5 seconds. Discuss the various strategies that you and your classmates come up with. How does your estimate relate to the graph?

P17.8 A conical reservoir is 12 meters deep and 8 meters in diameter.

(a) Confirm that the surface area of the water is described by $A(h) = \dfrac{\pi h^2}{9}$, given that the water is h meters deep.

(b) Complete the table below in a graphing app and construct a graph of surface area versus depth of water.

Depth (h)	0	3	6	9	12
Surface Area (A)					

(c) Estimate the volume of the reservoir using the points in the table in part (b) to *accumulate* surface areas for various depths. You may want to consider that the cone can be approximated with a stack of cylinders. How can you get a better estimate? Discuss your strategy.

(d) There is a well-known formula for the volume of a cone, so you can get an exact volume to compare with your estimates. What do your estimates, as well as the exact value, have to do with the graph of surface area versus depth?

P17.9 From midnight to 9 am, snow accumulates on a driveway at a rate modeled by the function $s(t) = 7\,t\,e^{\cos t}$, where t is in hours and s is in cubic feet per hour.

(a) Graph of $s(t)$. At about what time is the snow accumulating the fastest?

(b) Use your graph to come up with a rough estimate of the average rate of snow accumulation, and thus estimate the total amount of snow that accumulates by 9 am. Explain how this method relates to the graph of s versus t.

(c) Refine your estimate of the total amount of snow by using the rate of accumulation at each hour (1 am, 2 am, and so on) to estimate the amount of snow that accumulates during each hour. Your estimate of the total will be a sum of these nine numbers.

(d) Further refine your estimate by using half-hour intervals of accumulation.

(e) If you continue to use smaller intervals–say minute intervals, then seconds, etc.– to calculate the accumulation of snow, is there a theoretical limit to your sum? How does this theoretical limit relate to the graph of s versus t?

P17.10 The derivative of a step function has only one value, and yet it could be misleading to simply say that the derivative is a "constant function." Why?

P17.11 There is no general method that will find an explicit solution for every differential equation, but the antidifferentiation approach can be applied to separable equations. For example, you have seen how $\dfrac{dy}{dt} = -0.4\sqrt{y}$ can be solved by rewriting it as $\dfrac{1}{\sqrt{y}}\dfrac{dy}{dt} = -0.4$, then antidifferentiating both sides to obtain $2\sqrt{y} = -0.4t + C$. Notice that there are infinitely many solutions, thanks to the *antidifferentiation constant* C. Another separable example that occurs often is illustrated by $\dfrac{dy}{dt} = -0.4y$. You already know that a solution to this equation is an exponential function (which one?), but we can also use the separable technique. We can rewrite the equation as $\dfrac{1}{y}\dfrac{dy}{dt} = -0.4$, then antidifferentiate both sides to obtain $\ln y = -0.4t + C$, which is equivalent to $y = ke^{-0.4t}$.

(a) Solve $\dfrac{dy}{dt} = y^2 \sin t$ by the separable technique.

(b) Solve $\dfrac{dy}{dt} = y + 2$ by the separable technique.

(c) Show that $\dfrac{dy}{dt} = y + 2t$ is not a separable equation.

P17.12 (Continuation) Is it necessary to place an antidifferentiation constant on both sides of an antidifferentiated equation? Explain.

P17.13 A *cycloid* is traced out by the parametric equation $(x, y) = (t - \sin t, 1 - \cos t)$.

(a) Confirm graphically that one arch of this curve is traced out by using t-values from 0 to 2π.

(b) Estimate the area under one arch of the cycloid by three different methods:

- the average of the areas of a rectangle and a triangle
- the sum of the areas of five rectangles with heights on the curve
- using an antiderivative to find the area under a parabola that contains the peak and the x-intercepts of the arch.

(c) Which of the estimates in (b) do you think is closest to the true area?

P17.14 The integral $\int_0^{\pi} x \sin x \, dx$ is not one we can compute using the techniques we know now by finding an antiderivative. We can, however, estimate the area under the curve $y = x \sin x$ from $x = 0$ to $x = \pi$ by a variety of methods, a few of which are outlined below.

(a) First, divide the interval from $x = 0$ to $x = \pi$ into five subintervals using the sequence of points $0, \pi/5, 2\pi/5, 3\pi/5, 4\pi/5, \pi$. The first subinterval is $[0, \pi/5]$, the second subinterval is $[\pi/5, 2\pi/5]$, and so on up to the fifth subinterval $[4\pi/5, \pi]$.

(b) *Left-hand endpoints.* Approximate the area by summing up the five rectangles with heights determined by the value of $y = x \sin x$ at the left-hand endpoints of each subinterval. Draw a picture of the curve and the rectangles. How well do you think the sum approximates the actual area under the curve?

(c) *Right-hand endpoints.* Approximate the area by summing up the five rectangles with heights determined by the value of $y = x \sin x$ at the right-hand endpoints of each subinterval. Draw a picture of the curve and the rectangles. How well do you think the sum approximates the actual area under the curve?

(d) *Midpoints.* Approximate the area by summing up the five rectangles with heights determined by the value of $y = x \sin x$ at the midpoints of each subinterval. Draw a picture of the curve and the rectangles. How well do you think the sum approximates the actual area under the curve?

P17.15 (Continuation) Evaluate the integral $\int_0^{\pi} x \sin x \, dx$ using a numerical integrator on a calculator or some other technology. Now discuss the accuracy of the approximations from the previous problem.

P17.16 (Continuation) The area approximation with rectangles improves as the number of rectangles increases, which means the width of each rectangle decreases. In the limit as the number of rectangles increases without bound, the sum of the areas of the rectangles is exactly equal to the area. Which endpoint technique is represented by each of the following limits? Explain.

(a) $\lim_{n\to\infty} \sum_{k=0}^{n-1} (x_k \sin x_k)\left(\frac{\pi}{n}\right)$, where $x_k = k \cdot \frac{\pi}{n}$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} (x_k \sin x_k)\left(\frac{\pi}{n}\right)$, where $x_k = k \cdot \frac{\pi}{n}$

P17.17 Show that $\int_1^2 \frac{1}{x}\,dx$ and $\int_5^{10} \frac{1}{x}\,dx$ have the same value.

P17.18 (Continuation) Find the exact value of $\int_b^{2b} \frac{1}{x}\,dx$.

P17.19 Find the area of the region bounded by $y = \frac{1}{1+x^2}$, the x-axis, and the lines $x = 1$ and $x = -1$.

P17.20 Find the area of the "triangular" region enclosed by $y = \cos x$, $y = \sin x$, and $x = 0$.

P17.21 Recall that the Gini index is defined as twice the area of the region between the line $y = x$ and the Lorenz curve from $x = 0$ to $x = 1$. Suppose the equation of the Lorenz curve for the U.S. income distribution in 1990 is $L(x) = x^{2.36}$. Use $L(x)$ to calculate the value of the U.S. Gini index for 1990.

P17.22 The following definite integrals have something significant in common which will allow you to evaluate each of them by finding an antiderivative.

(a) $\int_0^3 \frac{2x}{1+x^2}\,dx$ **(b)** $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

(c) $\int_{-1}^0 \frac{e^x}{3+e^x}\,dx$ **(d)** $\int_4^5 \frac{2x+3}{x^2+3x-4}\,dx$

P17.23 Find the area of the region enclosed by the line $y = x$ and the parabola $y = 2-x^2$. You will need to find the points of intersection of the line and the parabola.

P17.24 The acceleration due to gravity g is not constant — it is a function of the distance r from the center of the Earth, whose radius is $R = 6.378 \times 10^6$ m, and whose mass is $M = 5.974 \times 10^{24}$ kg. You learn in physics that

$$g = \begin{cases} GMrR^{-3} & \text{for} \quad 0 \le r < R \\ GMr^{-2} & \text{for} \quad R \le r \end{cases}$$

where $G = 6.673 \times 10^{-11}$ Nm^2/kg^2 is the gravitational constant.

(a) Sketch a graph of g as a function of r.

(b) Is g a continuous function of r?

(c) Is g a differentiable function of r?

(d) What familiar value of g is found at the surface of the earth, where $r = R$?

Prelaboratory 17: Geometric Probability

PL17.1 (Random sum problem) Two real numbers are chosen at random between 0 and 10. What is the probability that the sum of the two numbers is less than 5?

(a) Represent the two numbers by x and y. Sketch the region that represents all possible ordered pairs (x, y) for this problem where $0 \leq x \leq 10$ and $0 \leq y \leq 10$.

(b) In the graph of part (a), which region represents the sum $x + y$ is less than 5?

(c) Use the ratio of areas in parts (a) and (b) to calculate the probability $P(x + y < 5)$.

(d) The process of modeling probability with areas is a technique known as Geometric Probability. Use this technique to calculate $P(x + y \geq 10)$.

PL17.2 (Random product problem) Two real numbers are chosen at random between 0 and 2. What is the probability that the product of the two numbers is less than 2?

(a) Represent the two numbers by x and y. Sketch the region that represents all possible ordered pairs (x, y) for this problem where $0 \leq x \leq 2$ and $0 \leq y \leq 2$

(b) In the graph of part (a), which region represents the product xy is less than 2?

(c) Use the ratio of areas in parts (a) and (b) to calculate the probability $P(xy < 2)$. You will need to use an integral to calculate part of the relevant area.

Laboratory 17: Geometric Probability

L17.1 Two real numbers are chosen at random between -2 and 2.

(a) Sketch the region of the xy-plane that symbolizes all possible pairs of numbers chosen. What is the area of this region?

(b) What is the probability that the sum of the squares of the two numbers x^2+y^2 is greater than 1? Sketch the region representing the solution to this problem and calculate the probability.

L17.2 Two real numbers are chosen at random between 0 and 10.

(a) Sketch the region of the xy-plane that symbolizes all possible pairs of numbers chosen. What is the area of this region?

(b) What is the probability that the quotient x/y of the two numbers is less than 1? Sketch the region representing the solution to this problem and calculate the probability.

(c) What is the probability that the quotient x/y of the two numbers is less than 10? Sketch the region representing the solution to this problem and calculate the probability.

(d) What is the probability that the quotient x/y of the two numbers is less than 1/2? Sketch the region representing the solution to this problem and calculate the probability.

L17.3 Two real numbers are chosen at random between 0 and 10.

(a) What is the probability that the sum of the two numbers is less than 5? Sketch the region representing the solution to this problem and calculate the probability.

(b) What is the probability that the sum of the two numbers is less than 15? Sketch the region representing the solution to this problem and calculate the probability.

L17.4 Two real numbers are chosen at random between 0 and 10.

(a) What is the probability that the product of the two numbers is more than 10? Sketch the region representing the solution to this problem and calculate the probability.

(b) What is the probability that the product of the two numbers is less than 20? Sketch the region representing the solution to this problem and calculate the probability.

(c) What is the probability that the product of the two numbers is more than 10 AND less than 20? Sketch the region representing the solution to this problem and calculate the probability.

L17.5 After you and your group have worked out your answers to the problems in this lab, turn in your results. Did you use calculus to solve all the problems? What type of geometric probability problem requires the use of calculus?

P18.1 The Fundamental Theorem of Calculus states that an integral $\int_a^b f(t)dt$ can be evaluated by the formula $F(b)-F(a)$, where F is an antiderivative for f. One of the issues we have glossed over is the assumption that an antiderivative of f actually exists. What must be true of the function f in order to apply the Fundamental Theorem? Can F be any antiderivative of f? Explain.

P18.2 (Continuation) Because of this close connection between integration and differentiation, integral signs are often used to denote antiderivatives. Thus statements such as $\int x^2\,dx = \frac{1}{3}x^3 + C$ are common, where C is called the constant of integration. The notation shown is a helpful way to order the steps in calculating a definite integral:

$$\begin{aligned}\int_5^8 x^2\ dx &= \frac{1}{3}x^3\bigg|_5^8 \\ &= \frac{1}{3}\cdot 8^3 - \frac{1}{3}\cdot 5^3 \\ &= 129\end{aligned}$$

(a) Why is there no need to insert a constant of integration in the antiderivative?

(b) Evaluate $\int e^{2x}\,dx$ and $\int_{-1}^{1} e^{2x}\,dx$.

P18.3 Suppose a projectile is propelled vertically into the air such that the height (in meters) as a function of time (in seconds) is given by $h(t) = -5t^2 + 1000t$.

(a) How long does it take for the projectile to return to the ground?

(b) What is the displacement of the projectile during the time it is in the air?

(c) What is the total distance traveled by the projectile?

P18.4 (Continuation) Compare the displacement to the distance traveled for an object moving with velocity $v(t) = \sin t$ from $t = 0$ to $t = 2\pi$.

P18.5 Evaluate each definite integral

(a) $\int_1^e \frac{1}{x}\,dx$ **(b)** $\int_{-2}^{2} |x|\,dx$ **(c)** $\int_0^{2\pi} \cos x\,dx$ **(d)** $\int_{-1}^{1} \left(e^x + e^{-x}\right)\,dx$

P18.6 Find the area of the region enclosed by the line $y = x + 2$ and the parabola $y = 4 - x^2$.

P18.7 Determine what property of a function f guarantees each of the following and graph a typical function in each case.

(a) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ holds for all a?

(b) $\int_{-a}^{a} f(x)\,dx = 0$ holds for all a?

P18.8 *S-I-R model*. The spread of an infectious disease can be modeled by differential equations that give the rate of change with respect to time of three segments of a population: S is the number of susceptibles who are not yet infected but could become infected; I is the number of people who are currently infected; and, R is the number of recovered people who have been infected but can no longer infect others or be reinfected. Consider the following set of coupled differential equations where a and b are positive constants:

$$\frac{dS}{dt} = -aSI \qquad \frac{dI}{dt} = aSI - bI \qquad \frac{dR}{dt} = bI$$

(a) Why is the sum of these three equations equal to 0?

(b) Explain each term in the equations and why they make sense.

(c) Suppose the starting values of each segment of the population are $S_0 = 1000$, $I_0 = 10$, and $R_0 = 0$. On the same set of axes, sketch plausible graphs of S, I, and R versus time.

P18.9 Evaluate each of the indefinite integrals by finding an antiderivative.

(a) $\displaystyle\int \frac{2x+5}{x^2+5x-6}\,dx$ **(b)** $\displaystyle\int \cos 3t\,dt$ **(c)** $\displaystyle\int \sec^2 2\theta\,d\theta$

P18.10 Check that $F(t) = \frac{1}{2}(t - \sin t \cos t)$ is an antiderivative for $f(t) = \sin^2 t$. Then find an antiderivative for $g(t) = \cos^2 t$.

P18.11 Jordan is jogging along a straight path with velocity given by a differentiable function v for $0 \le t \le 40$. The table below shows certain values of $v(t)$, where t is in minutes and $v(t)$ is measured in meters per minute. As always, you should have a visual representation before you as you solve this problem.

t (minutes)	0	12	20	24	40
v (m/min)	0	200	240	-220	150

(a) Explain the meaning of the definite integral $\displaystyle\int_0^{40} |v(t)|\,dt$ in this context.

(b) Find an approximate value of $\displaystyle\int_0^{40} |v(t)|\,dt$ using the given data.

P18.12 Give two reasons why $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ should be true.

P18.13 The area of the unit circle is given by the integral $\displaystyle 2\int_{-1}^{1} \sqrt{1-x^2}\,dx$. Explain. Use a geometric interpretation to find the value of this integral.

P18.14 For the differential equation $\dfrac{dy}{dx} = x - y$,

- **(a)** obtain a graph of the slope field;
- **(b)** add to the slope field a graph of the solution curve containing the point $(0, 1)$;
- **(c)** use Euler's method with 10 steps from $(0, 1)$ to find the value of the solution at the point with $x = 2$;
- **(d)** describe the behavior of the various types of solution curves as revealed by the slope field.

P18.15 The velocity of an object moving on the x-axis is $v(t) = \dfrac{1}{2}t^3 - 3t^2 + 10$.

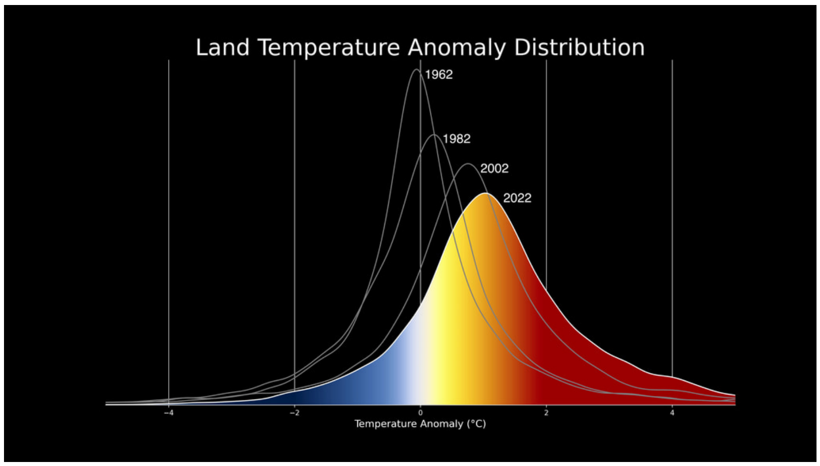
- **(a)** Find the displacement of the object from $t = 0$ to $t = 6$. How is this value related to area?
- **(b)** Write a single integral that represents the total distance traveled from $t = 0$ to $t = 6$. Find the value of this integral, for which you may want to use a numerical integrator.
- **(c)** What must be true of the graph of the velocity if the displacement over an interval is exactly 0?

P18.16 Show that the derivative with respect to x of $\ln|x|$ equals $\dfrac{1}{x}$, i.e. $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x}$. Thus show that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ is a true statement.

P18.17 Write a paragraph, along with helpful graphics, explaining the relationships between Euler's method, accumulating rate of change, a definite integral, and the sum of the areas of rectangles.

Prelaboratory 18: The Normal Curve

The following graph shows the distribution of temperature differences, or anomalies, for the years 1962, 1982, 2002, and 2022. Temperature anomalies are defined relative to the average global temperature over the baseline period 1951-1980. These curves are examples of what is known as a normal curve, which is a visualization of data that has a *normal distribution*.



NASA's Scientific Visualization Studio

PL18.1 Describe the main characteristics of a normal curve: shape, center, and spread.

PL18.2 Each of the curves for temperature anomalies has an area under the curve equal to 1. This allows us to interpret the graphs as probabilities without dividing by the total area as we did with geometric probability. Using the 1962 curve as a comparison, what do you observe about the transformations of the other 3 curves that keep the area equal to 1?

PL18.3 The standard normal curve has equation:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(a) Graph this function in an appropriate window that shows the essential features of the curve.

(b) What domain did you use in your graphing window?

(c) Show that the area under the curve is 1 by finding the value of the integral:

$$\int_{-k}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

where k is a large enough value to cover the domain of your graphing window. You will need to use a numerical integration tool on a calculator. (Some calculators have an infinity symbol that can be used in place of k.)

Laboratory 18: The Normal Curve

L18.1 Graph the following normal curve function on Desmos (or equivalent graphing tool). The parameters m and s should be defined as sliders.

$$f(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2}$$

(a) Start with $m = 0$ and $s = 1$. Confirm that your graph is the shape of the standard normal curve and that the area under the curve is 1.

(b) What is the effect on the graph of varying the value of m? How does this fit with your understanding of transformations of graphs? This parameter is known as the *mean* of the normal distribution.

(c) What is the effect on the graph of varying the value of s? How does this fit with your understanding of transformations of graphs? This parameter is known as the *standard deviation* of the normal distribution.

(d) Confirm that the area under the curve remains 1 no matter the values of m and s.

L18.2 Determine the values of the mean and standard deviation to fit a normal curve to the temperature data from Prelab #1 for

(a) 1962

(b) 2022

and show the graphs on the same axes.

L18.3 For the graphs in L18.2,

(a) How much has the mean temperature anomaly shifted between 1962 and 2022?

(b) Using the values of m and s for 2022 from L18.2, confirm the following integral calculations:

- the proportion of temperature anomalies greater than 0:

$$\int_0^\infty \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2} dx$$

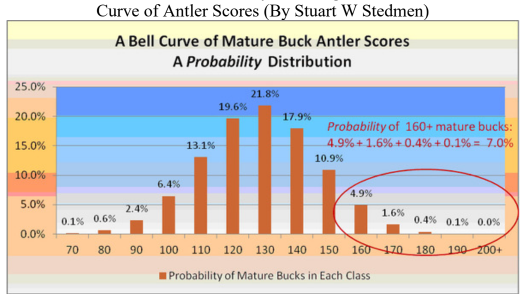
- the proportion of temperature anomalies within 1°C of 0:

$$\int_{-1}^1 \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2} dx$$

(c) In 1962, what percentage of the temperatures were more than 4°C above the baseline average? What was the percentage in 2022? How many more times likely was an extreme temperature anomaly above 4°C in 2022 compared with 1962?

L18.4 The following graphs are examples of data that is distributed normally. Determine the mean and standard deviation that fits a normal curve to each graph, then graph the normal distribution function for each.

The Bell Curve of Mature Buck Antler Scores: When you Manage a Buck Herd, You Manage a Bell Curve of Antler Scores (By Stuart W Stedmen)



https://igorsciencedotorg.files.wordpress.com/2013/01/criminal2.png

Heights of 3000 adult English male criminals, measured around 1900

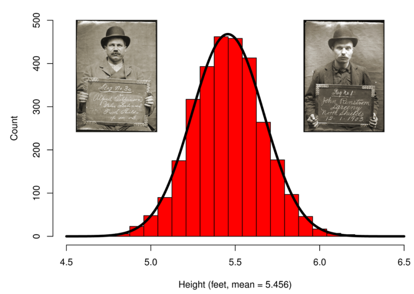
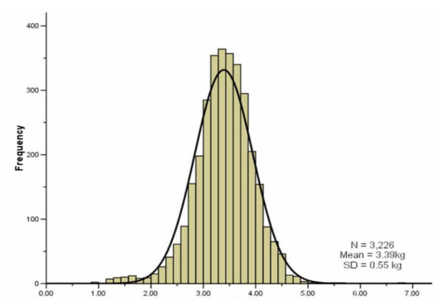


Figure 1 Distribution of birth weight in 3,226 newborn babies (data from O' Cathain et al 2002)



L18.5 Using the results of L18.4, what is probability of a birthweight between 2.5 and 4.5 kg?

L18.6 Name three additional data sets that you think might be normally distributed. Explain your choices.

L18.7 For the standard normal curve

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

find the second derivative and determine for what x-values the graph is concave up, is concave down, and has a point of inflection.

L18.8

(a) For the standard normal curve, what percentage of the area under the curve is within 1 unit of the center $x = 0$? This involves calculating the value of the integral:

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

(b) Do the same calculations to find the area under the curve within 2 units of the center. Now calculate the area within 3 units of the center.

(c) How is the *empirical rule* 68-95-99.7 related to your calculations?

L18.9

(a) For the general normal distribution

$$f(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2}$$

show graphically or otherwise that the points of inflection are s units away from the mean m.

(b) Show by a visual, graphical transformation demonstration that the empirical rule can be interpreted in terms of 1, 2, and 3 standard deviations s from the mean m.

L18.10 In addition to your answers to the problems above, submit a summary that includes the following concepts: what is a normal curve, what is a normal curve used for, what is the equation of the normal curve, what are the characteristics of the curve (center, spread, empirical rule), how are probability questions answered for a normal distribution.

P19.1 Water is being pumped into a pool at a rate $P(t)$ cubic feet per hour. The table below gives values for $P(t)$ for selected values of t.

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63

(a) Draw a graph of the ordered pairs $(t, P(t))$. Use the areas of trapezoids to estimate the amount of water that was pumped into the pool during these 12 hours.

(b) Over these same 12 hours, water has been leaking from the pool at the rate of $L(t)$ cubic feet per hour, where $L(t) = 25e^{-0.05t}$. How much water has leaked out during the time interval $0 \le t \le 12$?

(c) Use your answers to parts (a) and (b) to estimate the amount of water in the pool at time $t = 12$ hours.

P19.2 You have recently (in Problem Set 18) evaluated a number of integrals of the form $\int_a^b \sqrt{u}\,du$. For example, $\int_1^e \frac{1}{x}\sqrt{\ln x}\,dx$ is equal to $\int_0^1 \sqrt{u}\,du$, where $u = \ln x$. Notice that $u = 1$ corresponds to $x = e$ and $u = 0$ corresponds to $x = 1$, which is why the limits on the two integrals are different. Make up an example that is equivalent to $\int_4^9 \sqrt{u}\,du$, by replacing u by an expression $f(x)$ of your choice. Choose corresponding integration limits for your dx integral, then find its value. This technique of transforming one integral into another that has the same value is known as *integration by substitution*.

P19.3 Use integration by substitution to find the values of the following integrals.

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

(b) $\displaystyle\int_{2/3\pi}^{2/\pi} \frac{\cos(\frac{1}{x})}{x^2}\,dx$

P19.4 (Continuation) What area problem is represented by the integral $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$?

P19.5 The arc $y = \sqrt{x}$ for $0 \le x \le 4$ is revolved around the x-axis, which generates a surface known as a *paraboloid*. Note that the cross-sections perpendicular to the x-axis are circles.

(a) What is the area of the cross-section at $x = 1$? At $x = 4$? At $x = r$?

(b) Obtain and evaluate an integral for the volume of this paraboloid.

P19.6 Find the area bounded by the curves with equations $y = x^2 - 6x + 11$ and $y = -\frac{1}{2}x^2 + 2x + 4$.

P19.7 *The Mean-Value Theorem.* Suppose that f is continuous on the interval $a \le x \le b$, and that f is differentiable on the interval $a < x < b$. Draw the graph of f, and then draw the segment that joins $P = (a, f(a))$ to $Q = (b, f(b))$. Now consider all the lines that can be drawn tangent to the curve $y = f(x)$ for $a < x < b$. It is certain that at least one of these lines bears a special relationship to segment PQ. What is this relationship?

P19.8 The radius of a spherical container is r cm, and the water in it is h cm deep. Use an integral $\displaystyle\int_0^h A(z)\,dz$ to find a formula for the volume of water in the container. Check your formula for the special cases $h = 0$, $h = r$, and $h = 2r$.

P19.9 Explain the geometric significance of the result $\displaystyle\int_0^R 4\pi x^2\,dx = \frac{4}{3}\pi R^3$. In particular, contrast the way in which volume was accumulated in the preceding exercise with the way in which volume is being accumulated in this example.

P19.10 An American roulette wheel consists of slots numbered 1 to 36, half of which are red and half of which are black, plus two green slots numbered 0 and 00. The game is played by rolling a ball into the spinning wheel, where it comes to rest in one of the 38 slots.

(a) If you place a \$1 bet on red, and the ball ends up in a red slot, you win \$1 (meaning you get back your \$1 bet plus \$1). What is the probability of winning when betting on red?

(b) If you play red 1000 times, how much will you expect to win (or lose)? What is the average amount you win or lose per bet? This is known as the expected value of the game.

(c) Show that the expected value of playing red is the sum of two terms, each of which is the product of a probability and an amount won or lost.

(d) Why is it that casino games generally have a negative expected value from the perspective of the player?

P19.11 The conclusion of the Mean-Value Theorem does not necessarily follow if f is not known to be a differentiable function. Provide an example that illustrates this remark.

P19.12 The special case of the Mean-Value Theorem that occurs when $f(a) = 0 = f(b)$ is called *Rolle's Theorem.* Write a careful statement of this result. Does "0" play a significant role?

P19.13 Sketch a graph and shade the indicated region for each of the following integrals. Find the value of each integral, and give an explanation of why some of the answers are positive and some are negative.

(a) $\displaystyle\int_0^2 \left(x^2+1\right)\,dx$ **(b)** $\displaystyle\int_{-2}^0 \left(x^2+1\right)\,dx$

(c) $\displaystyle\int_0^{-2} \left(x^2+1\right)\,dx$ **(d)** $\displaystyle\int_0^1 \left(x^2-1\right)\,dx$

P19.14 Find the value of $\displaystyle\int_{-2}^2 \sin x\,dx$, and explain the answer.

P19.15 Given a function f, about which you know only $f(2)=1$ and $f(5)=4$, can you be sure that there is an x between 2 and 5 for which $f(x)=3$? If not, what additional information about f would allow you to conclude that f has this *intermediate-value property*?

P19.16 In the carnival game "Chuck-A-Luck" you roll three fair dice. If you roll three 6's you win \$5. If you roll two 6's you win \$3. If you roll one 6 you win \$1, and if you roll no 6's you lose \$1.

(a) On a single play of the game, how many ways can you win \$5? How many ways can you win \$3? How many ways can you win \$1? How many ways can you lose \$1?

(b) Show that the expected value of this game is \$0.

P19.17 Obtain a graph of the region enclosed by the positive coordinate axes, the curve $y=e^{-x}$, and the line $x=a$, where $a>0$.

(a) Write an expression for the area of this region. For what value of a is the area equal to 0.9? equal to 0.999? equal to 1.001?

(b) The expression $\displaystyle\int_0^\infty e^{-x}\,dx$, which is defined as $\displaystyle\lim_{b\to\infty}\int_0^b e^{-x}\,dx$, is an example of an *improper integral*. What is its numerical value, and how is this number to be interpreted?

Prelaboratory 19: The Exponential Distribution

PL19.1 What is the area underneath the curve $y = e^{-x}$? (The phrase "underneath the curve" in this context means the area bounded by the curve and the x- and y-axes.) Represent the area with an integral and then evaluate the integral.

PL19.2 What is the area underneath the curve $y = e^{-kx}$? Your answer should be in terms of the parameter k.

PL19.3 Find the value of A that makes the area underneath the curve $y = Ae^{-kx}$ equal to 1. In other words, solve the following equation for A in terms of k:

$$\int_0^\infty Ae^{-kx}dx = 1$$

How are A and k related? Since the area under the curve is 1, this function can be used as a probability distribution function known as the exponential distribution.

PL19.4 In the previous problem set, we defined expected value for a probability distribution as the sum of all the products of the outcome times probability of that outcome. With a continuous probability distribution, we cannot multiply outcomes by probabilities (since there are an infinite number of outcomes in a continuous distribution). Instead, we use the integral:

$$\int_a^b x \cdot P(x)dx$$

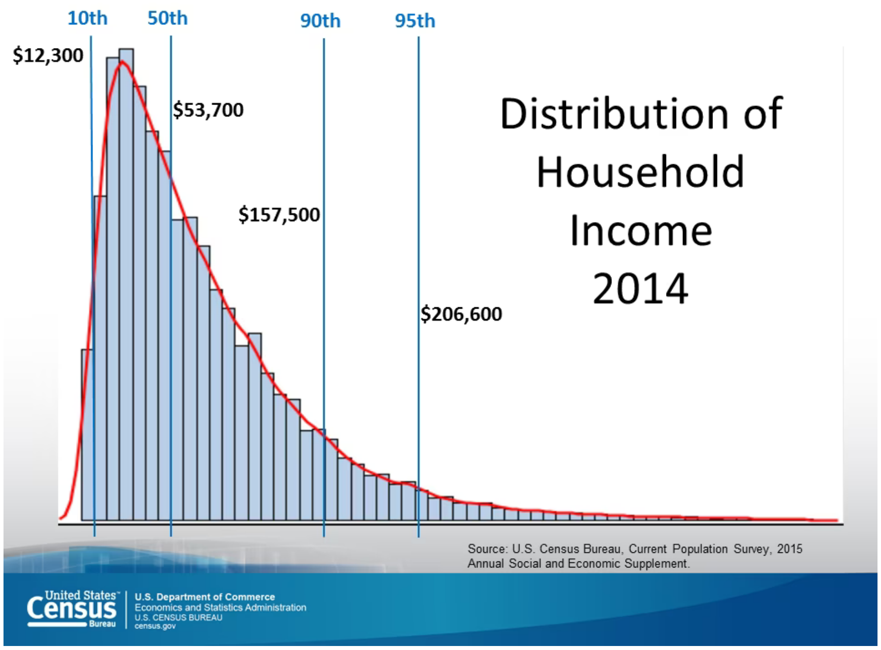
to represent the expected (or average) value of the distribution. For the normal distribution, the expected value is given by the integral:

$$\int_{-\infty}^\infty x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Show that the value of this integral is 0, and relate the answer to the graph of the normal distribution.

Laboratory 19: The Exponential Distribution

The following image is a graphic for income in the U.S. in 2014. Like many income distributions, this one looks similar to the exponential distribution.



https://www.census.gov/library/visualizations/2015/demo/distribution-of-household-income--2014.html

L19.1 Determine a reasonable exponential distribution ke^{-kt} that fits the data in the graphic.

(a) What is the area under the curve from $t = 0$ to $t = \infty$? How do you know?

(b) What is the meaning of the income at the 50th percentile? How well does your model fit this data point?

L19.2 Estimate from your model the proportion of households with incomes less than \$23,850, which was the poverty line for a family of 4 in 2014.

L19.3 Use an integral to estimate the percentiles for the following household income levels. How does \$50,000 relate to a proportion of the population? How does \$50,000 relate to a probability? Answer the same two questions for \$100,000.

(a) \$50,000

(b) \$100,000

(c) \$250,000

(d) \$500,000

L19.4 Explain why a distribution cannot be used to find the probability of a specific value (such as the probability of a household earning $75,560), but it can be used to determine the proportion of the population within a certain interval (such as incomes under $50,000). What proportion of households in 2014 earned between $100,000 and $250,000? Show the integral that leads to the answer.

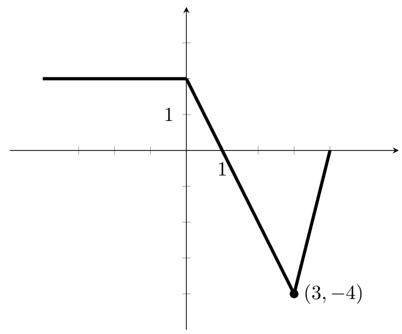
L19.5 Based on your model, use an integral with the exponential distribution to calculate the average household income in 2014. Recall that average value in this context can be found by summing up the products of t times the probability of an income of t occurring in the population for all possible values of t. Since the probability distribution is continuous, the result is an infinite sum that can be represented by the integral:

$$\int_0^\infty t \cdot ke^{-kt} dt$$

Compare this to the median income and explain the difference between the mean and median.

L19.6 Turn in a report with your answers to all the questions in the prelab and the lab.

P20.1 Let $f(t)$ be defined by the graph shown below. Let $g(x) = \int_0^x f(t)\,dt$ define an area function.



(a) Rewrite the following integrals in terms of g:

i. $\int_0^1 f(t)\,dt =$ **ii.** $\int_0^{-2} f(t)\,dt =$ **iii.** $\int_1^4 f(t)\,dt =$

(b) Define the area functions $h(x) = \int_{-3}^x f(t)\,dt$ and $k(x) = \int_1^x f(t)\,dt$. Fill in the chart below.

x	-3	-1	0	1	3	4
$g(x)$	-6					
$h(x)$		4				
$k(x)$			-1			

(c) Use the chart to sketch graphs of the three area functions on the same set of axes. What do you notice about the three graphs? Express the functions $h(x)$ and $k(x)$ in terms of $g(x)$.

(d) We previously learned that a function has infinitely many antiderivatives, each differing by a constant. How does this relate to the results in part (c)?

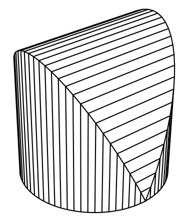
P20.2 Kelly completed a 250-mile drive in exactly 5 hours — an average speed of 50 mph. The trip was not actually made at a constant speed of 50 mph, of course, for there were traffic lights, slow-moving trucks in the way, etc. Nevertheless, there must have been at least one instant during the trip when Kelly's speedometer showed exactly 50 mph. Give two explanations for why this assertion is true — one using a distance-versus-time graph, and the other using a speed-versus-time graph. Make your graphs consistent with each other.

P20.3 (Continuation) A student drew the line that joins $(0, 0)$ to $(5, 250)$, and remarked that any actual distance-versus-time graph has to have points that lie above this line and points that lie below it. What do you think of this remark, and why?

P20.4 (Continuation) Another student was thinking that the area between the distance-versus-time graph and the time axis was a significant number. What do you think of this idea, and why?

P20.5 Sasha took a wooden cylinder and created an interesting sculpture from it. The finished object is 6 inches tall and 6 inches in diameter. It has square cross-sections perpendicular to the circular base of the cylinder. Draw the top view and the two simplest side views of this sculpture. The volume of a thin slice of the object is approximately $A(x)\Delta x$, where $-3 \le x \le 3$. Explain.

(a) What is an equation for the base if you center it at the origin of the xy-plane? What is an equation for $A(x)$, the area of a square cross section?



(b) What is the volume of the object?

P20.6 *Integration by Parts* is the name given to the Product Rule for derivatives when it is used to solve integration problems. The first step is to convert $(f \cdot g)' = f \cdot g' + g \cdot f'$ into the form $f(x) \cdot g(x)\Big|_{x=a}^{x=b} = \int_a^b f(x)g'(x)\,dx + \int_a^b g(x)f'(x)\,dx$. Explain this reasoning, then notice an interesting consequence of this equation: If either of the integrals can be evaluated, then the other can be too. Apply this insight to obtain the value of

$$\int_0^{\pi} x \cos x \, dx.$$

P20.7 (Continuation) Use integration by parts to verify the equation

$$\int_0^{\infty} x \cdot ae^{-ax}\,dx = \frac{1}{a},$$

Do this by assigning $f(x) = x$ and $g'(x) = ae^{-ax}$. Notice that this is the expected value of the exponential distribution.

P20.8 Use integration by parts to find the following indefinite integrals.

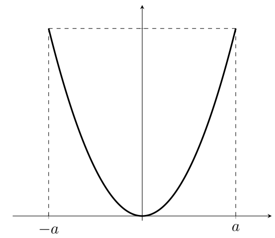
(a) $\int x \sin x\,dx$ **(b)** $\int te^t\,dt$ **(c)** $\int x \ln x\,dx$

P20.9 Evaluate the following improper integrals.

(a) $\int_1^{\infty} \frac{1}{x}\,dx$ **(b)** $\int_1^{\infty} \frac{1}{x^2}\,dx$ **(c)** $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$

P20.10 (Continuation) For what values of p does the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ have a finite value? Explain and justify your answer.

P20.11 The diagram shows the parabolic arc $y = x^2$ inscribed in the rectangle $-a \leq x \leq a$, $0 \leq y \leq a^2$. This curve separates the rectangle into two regions. Find the ratio of their areas, and show that it does not depend on the value of a.



P20.12 The area function $A(x) = \int_1^x \frac{1}{t}\,dt$ defines a well-known function. What is it? What value of x gives an area of 1, which is to say $A(x) = 1$? For what x is $A(x) = -1$?

P20.13 (Continuation) Explain graphically and algebraically why $A(b) - A(a) = \int_a^b \frac{1}{t}\,dt$.

P20.14 The temperature for a typical May 15$^{\text{th}}$ in Exeter can be modeled by the function $T(t) = -12.1\cos\left(\frac{\pi}{12}t\right) + 55.5$, where t is the hours after sunrise, and $0 \leq t \leq 15$. How do we find the average temperature during the nearly 15 hours of sunlight on this day?

(a) Suppose we use the temperature at the beginning of each hour to calculate the average. Find the value of $\frac{1}{15}\sum_{i=0}^{14} T(t_i)$, where $t_0 = 0$, $t_1 = 1$, ..., $t_{14} = 14$.

(b) You can obtain a more accurate average by using the temperatures at half-hour intervals. Calculate the value of $\frac{1}{30}\sum_{i=0}^{29} T(t_i)$, where $t_i = 0.5i$. What would this summation look like if we used the temperature at the end of each half hour interval?

(c) Confirm that the average temperature using n equally spaced times starting at $t_0 = 0$ is given by $\frac{1}{n}\sum_{i=0}^{n-1} T(t_i)$.

(d) The time interval used in the averaging process is $\Delta t = \frac{15}{n}$. Confirm that introducing Δt into the previous summation leads to $\frac{1}{15}\sum_{i=0}^{n-1} T(t_i)\Delta t$.

(e) The previous summation has the familiar form of a Riemann sum and an area using left-hand endpoint rectangles. As we let n increase without bound, the summation approaches an integral. Write out the definite integral represented by $\lim_{n\to\infty} \frac{1}{15}\sum_{i=0}^{n-1} T(t_i)\Delta t$.

(f) Calculate the value of this integral.

P20.15 (Continuation) *Average value.* Explain why the average value of a function over an interval $a \le x \le b$ can be defined by the limit of a Riemann sum

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=0}^{n-1} f(x_i)\Delta x, \quad \text{or equivalently} \quad \frac{1}{b-a}\int_a^b f(x)\,dx.$$

P20.16 The driver of a red sports car, which is rolling along at 110 feet per second, suddenly steps on the brake, producing a steady deceleration of 25 feet per second per second. How many feet does the red sports car travel while coming to a stop?

P20.17 To evaluate $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx$, Steph tried integration by parts. Was this a good idea?

P20.18 Find the average value of $\sin x$ over one arch of the curve, $0 \le x \le \pi$. Compare your answer to the average of the maximum and minimum values on this interval.

P20.19 The *Mean-Value Theorem for Integrals* says: If f is a function that is continuous for $a \le x \le b$, then there is a number c between a and b for which $f(c)\cdot(b-a) = \displaystyle\int_a^b f(x)\,dx$. Interpret this statement graphically, and relate it to the average value of $f(x)$ on the interval $[a, b]$.

P20.20 (Continuation) Use the Fundamental Theorem of Calculus to show how the Mean-Value Theorem for Integrals is related to the Mean-Value Theorem for Derivatives.

P20.21 For what type of velocity function will the average value of velocity, which can be written as $\displaystyle\frac{1}{b-a}\int_a^b v(t)\,dt$, equal the average of the initial and final velocities $\displaystyle\frac{1}{2}(v(a)+v(b))$? In other words, under what conditions will the following equation be true?

$$\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{2}(v(a)+v(b))$$

P20.22 Show that the expected value of the standard normal distribution is the center of the distribution. In other words, confirm the following equation:

$$\int_{-\infty}^{\infty} x\cdot\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = 0.$$

P20.23 What is the average value of an odd function over the interval $[-a, a]$?

P20.24 A model for the pH level of the saliva in the mouth t minutes after eating a piece of candy is given by $f(t) = 6.5 - \dfrac{20t}{t^2+36}$.

(a) What is the average pH in the mouth during the first 10 minutes after eating a piece of candy?

(b) During the first hour?

P20.25 Let $f(x) = x^2 - 2$. Investigate the area functions $A(x) = \displaystyle\int_0^x t^2 - 2\,dt$.

(a) Evaluate integrals to fill in the following table.

x	-2.0	-1.0	-0.5	0.0	0.5	1.0	2.0
$A(x)$							

(b) Plot the ordered pairs $(x, A(x))$.

(c) At what points do the minimum and maximum values of $A(x)$ occur on the interval $-2 \le x \le 2$? Where does $f(x)$ equal 0?

(d) What can you conclude about the functions $A(x)$ and $f(x)$?

P20.26 (Continuation) Use an antiderivative to show that the derivative of $\displaystyle\int_1^x t^2\,dt$ is x^2, which can be written symbolically as $\dfrac{d}{dx}\displaystyle\int_1^x t^2\,dt = x^2$.

P20.27 (Continuation) Show $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$. Hint: Assume a function F is an antiderivative of f. This result is often called the *Second Fundamental Theorem of Calculus.*

P20.28 (Continuation) We now have two Fundamental Theorems of Calculus. Explain how they are related.

In a population model in which a predator feeds on a prey as the primary food source, changes in the size of each population depend jointly on the number of individuals in each species. A system of differential equations for lynx (predator) and hares (prey) is:

$$\frac{dh}{dt} = a \cdot h - b \cdot h \cdot l \qquad\qquad \frac{dl}{dt} = -c \cdot l + d \cdot h \cdot l$$

where l represents the number of lynx, h represents the number of hares, and a, b, c, and d are constants. Solving this system symbolically is beyond this course, but we can generate approximate values for each population by using Euler's method.

Prelaboratory 20: Predator/Prey Model

PL20.1 In a population model for lynx and hares, the lynx (*predator*) feeds on the hare (*prey*). The changes in the size of each population depend on both populations because of the predator/prey interaction. The differential equation for the hare population is

$$\frac{dh}{dt} = 0.05h - 0.001hl,$$

and the differential equation for the lynx population is

$$\frac{dl}{dt} = -0.03l + 0.0002hl$$

.

(a) Write the Euler's method equations that approximate the values of the two populations, h_n and l_n, in terms of h_{n-1}, l_{n-1}, and the stepsize Δt. You do not need to compute any values.

(b) Explain the effect of each term in the two differential equations. In particular, consider which terms give the natural growth/decline rates in the absence of the other species and which terms are due to interaction between the species.

(c) Make a conjecture about how you think the growth and decline of one population depends on the other population.

Laboratory 20: Predator/Prey Model

L20.1 Write the recursive equations that use Euler's method to approximate the values of the two populations, h_n and l_n, in terms of h_{n-1}, l_{n-1}, the constants a, b, c, and d, and the stepsize Δt.

L20.2 Let the initial size of the hare population be 200 and the initial lynx population be 50. Use the following values for the four constants: $a = 0.05$, $b = 0.001$, $c = 0.03$, $d = 0.0002$. Examine numerically and graphically the populations versus time for 200 months with a stepsize of 1 month. Since the equations both involve values of h_{n-1} and l_{n-1}, you will need to generate the values of the two populations in sync, both populations one step at a time.

L20.3 Discuss the relative shapes of the two graphs for lynx and hares. It is helpful to graph both populations on the same set of axes. Are the graphs periodic? Are they sinusoidal? What is the relationship between the peaks and valleys of the two graphs? What seem to be the values about which each population oscillates?

L20.4 Examine the effect of small changes in the parameter values. For example, change a by a small amount and see what happens to the populations. How does the value of each parameter affect the graphs? What happens if you change the starting value of a population?

L20.5 The rate of change $\dfrac{dl}{dh}$ of the lynx with respect to the hares can be modeled by the ratio of $\dfrac{dl}{dt}$ to $\dfrac{dh}{dt}$, so that $\dfrac{dl}{dh} = \dfrac{\frac{dl}{dt}}{\frac{dh}{dt}}$. Using the values of the parameters from #L20.2, obtain a graph of the slope field for $\dfrac{dl}{dh}$. What does the slope field tell you about the relationship between l and h? Draw a particular solution curve in the slope field using the initial values of $l_0 = 50$ and $h_0 = 200$.

L20.6 Each of the differential equations in this system can be factored to find ordered pairs (l, h) that make the differential equations equal to 0. One such ordered pair is $(0, 0)$. What is another ordered pair with this property? These points are called stable points since the populations will remain constant if set to those values. How does this stable point that is not $(0, 0)$ relate to the graphs of the lynx and hares versus time, and also the slope field and solution curve for lynx verse hares?

L20.7 Assemble your work into a report that highlights with graphs and explanations the key concepts of this lab. Be sure to include how the techniques of calculus were used to investigate the predator/prey model.

Prelaboratory 21: Calculus and Data Analysis

PL21.1 The following table shows temperature in degrees F versus time in minutes for a cold drink that is taken out of the refrigerator at time 0.

Time (min)	Temperature (°F)
0	35.0
1	38.8
2	42.4
3	45.7
5	51.5
6	54.1
8	58.6
11	64.3

We know from previous work that a model for this data is an exponential curve that levels off at the ambient temperature. Enter this data in the spreadsheet of a Geogebra file, and then obtain a graph. You will notice that we do not have enough data to easily determine the ambient temperature directly from the graph.

PL21.2 Rather than trying to guess the ambient temperature, we can use our calculus knowledge to estimate parameters of an appropriate model for the data. A differential equation for the heating phenomenon is

$$\frac{dT}{dt} = k(A - T),$$

where A, the ambient temperature, and k, the growth rate, are the parameters of the model. Explain a rationale for this equation.

PL21.3 The differential equation above is linear in terms of the temperature T, which implies that the ordered pairs $\left(T, \dfrac{dT}{dt}\right)$ have a linear trend. We are not given values of $\dfrac{dT}{dt}$, but we can approximate each of the values with a difference quotient $\dfrac{\Delta T}{\Delta t}$. You should now add to your spreadsheet a column with the difference quotients.

Laboratory 21: Calculus and Data Analysis

L21.1 Each difference quotient approximates the derivative over an interval, so each difference quotient should be paired with the average T on each interval (rather then the left-hand or right-hand T). Make a column on the spreadsheet with the average T on each interval.

L21.2 Now you are ready to make a plot of the ordered pairs that approximate $\left(T, \dfrac{dT}{dt}\right)$ and fit a line to the data. Highlight the two columns with average T and with the difference quotients, then select the menu option for analyzing two-variable data. Be sure it shows the variables in the correct order. If not, there is an option to flip the ordered pairs. Choose the linear least-squares option, which will open a window with a plot of the ordered pairs, the least-squares line, and the equation of the line.

L21.3 The linear least-squares line has an equation of the form $y = mx + b$. Equate your least-squares equation with the differential equation above, and confirm that $m = -k$ and $b = kA$.

L21.4 Use the slope and intercept from the least-squares equation to solve for k and A.

L21.5 Now we are ready to solve the differential equation to find a model for the data. Show that the solution is $T(t) = A - Ce^{-kt}$, where C is a constant of integration.

L21.6 Substitute your values for A and k into the model $T(t)$. The value of C can be estimated from the initial value of the temperature. Graph your model along with the original data for time and temperature.

L21.7 Discuss the various characteristics of your model. What is the ambient temperature? What is the meaning of k? How well does your model fit the data?

Part II: Atmospheric Carbon Dioxide

L21.8 Open the Geogebra file "CO2 data" for the annual average atmospheric CO_2 concentration (in parts per million) since 1950. Obtain a graph of the ordered pairs (Years since 1950, CO_2).[1]

L21.9 An exponential model looks promising for this data, so try to fit an exponential least squares model to the ordered pairs (Years since 1950, CO_2). You will undoubtedly notice that it does not give a good result. This is because an exponential least squares model can be used only if the data has a horizontal asymptote at 0. This data has a horizontal asymptote well above 0, which represents the ambient CO_2 concentration from before the industrial revolution. As with Parts I and II, we will use a differential equation model to estimate the unknown parameter for ambient CO_2 concentration.

[1] http://www.esrl.noaa.gov/gmd/ccgg/trends/data.html

L21.10 Explain why the following differential equation is a reasonable model for exponential growth above an ambient level A, where k is a growth constant and C is the atmospheric CO_2 concentration at time t:

$$\frac{dC}{dt} = k(C - A).$$

L21.11 This differential equation is linear with respect to C, so using the techniques learned in this lab, find the least squares line that fits the ordered pairs $\left(C, \dfrac{dC}{dt}\right)$.

L21.12 Use the slope and intercept from the least squares line to estimate values for k and A.

L21.13 Confirm that the solution to the differential equation is $C(t) = A + Be^{kt}$, where B is a constant of integration, and t is years since 1950.

L21.14 Substitute values for k and A into your model, and use $C(0)$ to estimate a value for B. Graph your model together with the CO_2 data.

L21.15 Discuss and critique your model. What does your model suggest about future levels of atmospheric CO_2? Is it reasonable to extrapolate in this way?

Part III: The Logistic Model for Growth

L21.16 In this section you will fit a logistic model to the data for the total number of Starbucks stores open at the end of each year from 1987 through 2008. Open the Geogebra file named "Starbucks data" to find the data in the spreadsheet view.[2] To make the data more manageable, insert a column for "Years since 1987" by subtracting 1987 from the Year column. Now obtain a plot of the set of ordered pairs (Years since 1987; Stores).

L21.17 Discuss why a logistic model is a reasonable model for this data.

L21.18 One of the parameters for a logistic model is the maximum number of stores. The graph of the given data does not show this growth limit, but as in Part I, we can use a differential equation to estimate this parameter. Our differential equation for logistic growth is

$$\frac{dS}{dt} = kS\left(1 - \frac{S}{M}\right),$$

where S is the number of stores as a function of time t, k is a growth rate, and M is the upper limit on the number of stores. We can rewrite this equation as

$$\frac{1}{S}\frac{dS}{dt} = k\left(1 - \frac{S}{M}\right),$$

which is a linear equation with ordered pairs $\left(S, \dfrac{1}{S}\dfrac{dS}{dt}\right)$. Create a column in your spreadsheet for the average S on each interval (why?) and then a column for the difference quotients $\dfrac{\Delta S}{\Delta t}$ divided by the average S.

[2] http://www.starbucks.com/about-us/company-information/starbucks-company-timeline

L21.19 Make a plot of the ordered pairs that approximate $\left(S, \dfrac{1}{S}\dfrac{dS}{dt}\right)$ and fit a line to the data using the menu options as in Part I. You may need to go to the menu Options, Rounding, and change the rounding default to five significant figures so that the value of the slope is displayed.

L21.20 Equate your least-squares equation with the differential equation above, and confirm that $b = k$ and $m = -\dfrac{k}{M}$. Use the slope m and intercept b from the least-squares equation to solve for k and M.

L21.21 The solution to the logistic differential equation is

$$S(t) = \frac{Ce^{kt}}{1 + \dfrac{C}{M}e^{kt}},$$

where C is a constant of integration. (You do not need to confirm this solution.) Substitute your values for M and k into this model. The value of C can be approximated by the initial number of stores. Graph your model along with the original data for the ordered pairs (Years since 1987, number of stores).

L21.22 Discuss various characteristics of your model. What is maximum number of stores predicted by your model? How well does your model fit the data?

L21.23 Add the following data points to your graph, and discuss how well your prediction based upon data through 2008 fits the data that comes after 2008.

Year	2009	2010	2011	2012	2013	2014
Stores	16 635	16 858	17 003	18 066	19 767	21 366

Summary

L21.24 Prepare a report with a summary of your results for each part, including data, graphs, how parameter values were determined, and an evaluation of your model. What calculus techniques are common to the three parts? What general problems are you now able to solve with data analysis and calculus that you were unable to solve prior to this lab?

In this lab we will explore the SIR model for infectious diseases, a model that can be investigated using our knowledge of differential equations and numerical integration using Euler's method.

Prelaboratory 22: The SIR Model

PL22.1 The SIR Model for Spread of Disease:

S = susceptible proportion of population
I = infected proportion of population
R = removed proportion of population

- Individuals move from I to R at a rate proportional to I, the number of infected people
- Individuals move from S to I at a rate jointly proportional to S and I
- Assume the population P consists of S + I + R, and that the total remains constant

Explain why the assumptions above are reasonable for this mathematical model.

PL22.2 The SIR Model in Differential Equations Form:

$$\frac{dS}{dt} = -\beta \cdot S \cdot I$$

$$\frac{dI}{dt} = \beta \cdot S \cdot I - \alpha \cdot I$$

$$\frac{dR}{dt} = \alpha \cdot I$$

Show from these equations that $\frac{dP}{dt}$ equals 0. Why is that to be expected?

PL22.3 We can solve this system numerically using Euler's method with a step-size of 1. Write the recursive equations for this model.

PL22.4 Sketch a graph of what you think this model will look like as an outbreak runs through the population, starting with a small number of infected people. Put all three curves $S(t), I(t), R(t)$ on the same set of axes. Assume that $S(0) \approx 1, I(0) \approx 0$, and $R(0) = 0$. (Note that all three quantities are in units of proportions of the total population, so the range of values varies between 0 and 1, where 1 is 100% of the population.)

Laboratory 22: The SIR Model, Part 1

L22.1 Use a spreadsheet to generate values and a graph of $S(t), I(t)$, and $R(t)$ for $t = 0$ to 30 days using the equations from Prelab 3. Use parameter values $\beta = 0.8$ and $\alpha = 0.25$. Comment on the features of these graphs. How do your graphs compare with your prediction from Prelab 4?

L22.2 Interpretation of parameters:

- α is the recovery rate, which is the reciprocal of how many days someone is infectious and can transmit the disease. For COVID-19, the CDC has recommended a quarantine of 14 days, so we can estimate $\alpha = 1/14$.
- β is the number of infectious contacts per day between an infected person and a susceptible person. β is difficult to estimate and depends not only on the infectiousness of the disease but also on the amount of social mixing that regularly occurs.

What we do have an estimated value for is R_0, known as R naught, which is the typical number of people to whom an infected person will transmit the disease. This transmissibility parameter can be estimated as $\dfrac{\beta}{\alpha}$. Why does this make sense? Estimates for R_0 are generally between 3 and 4 (without social distancing). Let's use $R_0 = 3.5$. What estimate for β does this produce?

L22.3 Run the model (i.e. generate values for SIR) using the parameter estimates in the previous problem. Use an initial infection rate of 0.01 (which is 1%), and run the model for 100 days. Comment on your results.

L22.4 Run the model with progressively smaller values of $I(0)$, the initial number of infected. Examine what occurs with $I(0) = 0.001, 0.0001$, and 0.00001. (In some cases, you may need to run the model for more than 100 days.) What is the overall effect of the different numbers for the initial infection?

L22.5 *Flattening the Curve*: The effect of social distancing is to lower the value of R_0, hopefully below 1, which will lead to a net decline in infected people over time. Which parameter is influenced by social distancing? Determine a value that makes R_0 equal to about 0.7. Run the model with a parameter change for social distancing introduced during the exponential rise part of the $I(t)$ curve. What happens to the curve? What happens later if you relax social distancing and R_0 goes back to a number greater than 1?

Laboratory 22: The SIR Model, Part 2

In this activity we will continue our exploration of the SIR model for infectious diseases, using calculus to find the location of the maximum of the $I(t)$ curve and to gain insight into herd immunity.

L22.6 In the SIR model, use the derivative of $I(t)$ to find the maximum for $I(t)$ in terms of S.

L22.7 What will happen to the spread of infection if the susceptible people are less than a certain proportion of the population? What is that proportion given the parameter values we found in Part 1?

L22.8 A closer look at the contact number:

- $\alpha = 1/$(number of days someone is infectious)
- $\beta =$ number of infectious contacts per day for someone who is infected
- $\beta * \frac{1}{\alpha} =$ average number of people who will be infected by someone who is infected

Justify the following statement: This $\frac{\beta}{\alpha}$ which we called R_0 in Part 1, is also known as the contact number c, so if $S(0) < 1/c$, then there will be no outbreak of the disease.

L22.9 Epidemiologists are estimating that *herd immunity* for COVID-19 is about 60-70%. Where are those numbers coming from? How does herd immunity relate to the contact number? How does a population reach herd immunity? How is this process affected by *vaccine hesitancy*?

L22.10 By way of comparison, R_0 for measles is around 12 (although it is often cited as high as 18). What is herd immunity (and hence the vaccination target rate) for measles?

L22.11 A median value of R_0 equal to 5.7 was published early in the pandemic by researchers in Europe. What is the herd immunity for $R_0 = 5.7$?

L22.12 Demonstrate herd immunity by running the SIR model from Part 1 with the initial number of susceptible people at the proportion determined by herd immunity for $R_0 = 3.5$. Graph S and I on the same axes. What do you notice about the graph of I? Is there an outbreak of the disease? (Hint: adjust your vertical axis so that you can see changes in the graph of I.)

Goal: The goal of this activity is to practice applying your differentiation skills to data collected in the chemistry lab (most specifically to **determine an equivalency point**). It is assumed that you have no prior knowledge of chemistry and you are not responsible for any of the chemistry content beyond being prepared for class. The idea is that you will have a better sense of differentiation by applying your skills to data you physically collect?data you watch changing in real time.

Background: For the purposes of this lab, an acid can be thought of as a chemical substance that increases the concentration of H+ in solution. In our lab we will add sodium hydroxide (NaOH: a strong base) to vinegar ($HC_2H_3O_2$: a common weak acid). The OH– from the base will combine with the H+ from the acid to form water:

$$HC_2H_3O_2\,_{(aq)} + NaOH\,_{(aq)} \;\rightarrow\; H_2O\,_{(l)} + NaC_2H_3O_2\,_{(aq)}$$

The above reaction is known as a neutralization (water is neutral) and the physical act of slowly adding the base to the acid is an example of a titration.

As you are adding your base to the acid you will measure the pH of the solution. The function "p" is defined such that p(x) = −log(x). In our specific case pH = −log[H+]. A less acidic solution has less H+ and a more acidic solution has more H+. A glass of wine might have a small bit of H+ such that [H+] = 10^{-4} and pH = 4. Lemon juice (more acidic) might have enough H+ such that [H+] = 10^{-2} and pH = 2. Acids have pH values less than 7 and bases pH values greater than 7.

As you add the base (NaOH) to the acid ($HC_2H_3O_2$) the solution becomes less and less acidic. When the number of OH– ions added exactly matches the number of $HC_2H_3O_2$ molecules originally in the solution you have reached the equivalency point. This is an important point from a chemist's perspective and you should take a chemistry class if you wish to know more. The equivalency point will correspond to a maximum in the first derivative curve of pH versus volume — it is worth thinking about why this is the case both during and after the lab activity (it is a little early to worry about at this point).

Pre-Lab Questions:

1. As you add more and more base to your acid, will the pH increase or decrease? Explain.
2. A student measures the pH of human gastric acid (stomach acid) to be 2.5. What is the H+ concentration, [H+], in the gastric acid?
3. Human blood typically maintains an H+ concentration of [4.0×10^{-8}]. What is the pH of human blood? Is human blood acidic or basic?
4. May require research: What is numerical differentiation and more specifically what is Newton's difference quotient? Can Newton?s difference quotient be used to approximate a second derivative? Explain.

Procedure:

• Use a volumetric pipette to measure out 25.0 mL of $HC_2H_3O_{2\ (aq)}$ and transfer to a 150 mL beaker.

• Add a few drops of the indicator phenolphthalein. Phenolphthalein turns pink at the end-point of the titration.

• On the data table below, record the concentration of the sodium hydroxide written on the bottle.

• Use a volumetric pipette to measure out 25.0 mL of $HC_2H_3O_{2\ (aq)}$ and transfer to a 150 mL beaker.

• Add a few drops of the indicator phenolphthalein. Phenolphthalein turns pink at the end-point of the titration.

• On the data table below, record the concentration of the sodium hydroxide written on the bottle.

• With a waste beaker under the tip, pour some NaOH into the buret, allow some to drain out, and make sure there are no air bubbles in the tip of the buret. Fill the burette with the NaOH solution up past the 0 mL mark and then drain down to the 0.0 mL mark.

• Open up Logger Pro and make sure the Y-axis reads pH and the X-axis reads volume — see Mr. M if this is not the case.

• You may need to calibrate the pH probe — see any instructions on the overhead.

• Set the beaker containing the acetic acid solution on the stir plate under the buret, add the stir bar to the beaker, and place the pH probe in the beaker — when moving the pH probe from one solution to another or back to the storage solution, thoroughly rinse it with distilled or deionized water and gently pat-dry before final transfer.

• Turn on the stir plate and arrange everything so that the stir bar won't hit the pH probe. The pH meter can be set against one side of the beaker and out of the way of the stir bar (see picture) by hanging it over the edge using the storage solution cap and a rubber band (a partner can hold it there until you are sure it is steady or you can clamp it in place). Add enough distilled water to be sure the pH probe is covered.

• Begin the LoggerPro program by pressing COLLECT . BEFORE YOU BEGIN THE TITRATION . . . plot the initial pH of your acetic acid solution by pressing keep and then writing "0" when prompted for the volume. Note this initial pH in the data section below.

• Add 1.0 mL of NaOH and plot the new pH, again by pressing keep and then writing "1.0" as the program prompts you for a volume. Add successive 1.0 mL volumes of the NaOH and record the pH and **total volume** of NaOH added after each addition. After adding approximately 15ml **start adding only** 0.50 ml each time. After adding approximately 20ml **start adding only** 0.25 ml (or as close as you can) each time until you clear the equivalency point. Keep adding; you can return to 1.0 ml volumes when you have reached a high pH that has clearly leveled out (is not significantly changing).

• After you have added a lot of base (past equivalency), add the NaOH in one more 5.0 mL volume, keep that data point and hit STOP at a total volume of at least 25 mL NaOH added.

• When the titration is complete, hit STOP and then **save your data**.

• Now **export** your data: In logger pro select file, export data, CSV format. Save this file on a flash drive or on the cloud or email it to yourself — you need to be able to open it in an Excel spreadsheet on your own computer.

• See Mr. M **before** you print the graph (he will move the x axis **up** to create a "negative" region). Use file, "print graph" **not** print. Print four copies of your pH plot (2 per partner).

Data:

• Concentration of standardized NaOH used in the titration. [NaOH] =____________

• The pH of the acetic acid solution before the titration began. $pH_{HC2H3O2}$ = ____________

Analysis of the Data: (Complete the following in **Excel**)

• Using Newton's difference quotient and your original data, create a new data table representing a numerical approximation of the **first** derivative. Generate a plot of this data.

• Using Newton's difference quotient create another new data table representing a numerical approximation of the **second** derivative. Generate a plot of this data.

• **Print** a spreadsheet showing the original data, the first derivative data, and the second derivative data.

Questions and Analysis: (Complete the data analysis above in **Excel** before proceeding)

(1) What volume of added NaOH corresponds to the "equivalency point" in this titration? Explain your reasoning and all the evidence that points to this conclusion. Use calculus vocabulary whenever possible.

(2) On one of the original titration plots printed in class, graphically show a representation of Newton's difference quotient for any two consecutive points around x (volume NaOH added) = 10ml (or anywhere there is a nice curve). Explain how this differs from the "true" derivative? How could it be improved upon?

(3) On the other copy of the original titration plot printed in class, sketch a plot of an idealized first derivative. In a different color, sketch the plot of an idealized second derivative. Colored pencils are probably best to keep everything legible as you draw one sketch atop another.

(4) Referencing a rate, what does the maximum in the first derivative curve physically represent? (Be sure you are referencing a rate).

(5) Explain any points of interest in the second derivative curve.

(6) Titrations are usually only examined using numerical methods. Why do you think this is the case? Explain. Feel free to research.

What to turn in: (Do not type anything!)
(1) The answers to the pre-lab questions.
(2) The two titration plots you printed in class, with the appropriate markings as required in the "Questions and Analysis" section above.
(3) A printed spreadsheet showing the original data, the first derivative data, and the second derivative data.
(4) The answers to the each numbered item in the "Questions and Analysis" section above.

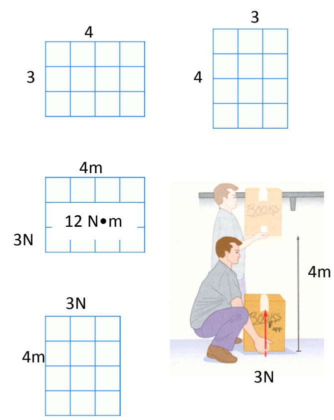
Extra Credit: The early part of our titration curve can be modeled using:

$$pH = pK_a + \log\left(\frac{x}{\text{original moles acid} \times 0.025L - x}\right)$$

Where x is the number of moles of OH? added (not ml). Using the second derivative of the above function find an inflection point in the early part of the curve. Convert this mole number to volume (ml) using the molarity of base solution. What do you notice about this value?

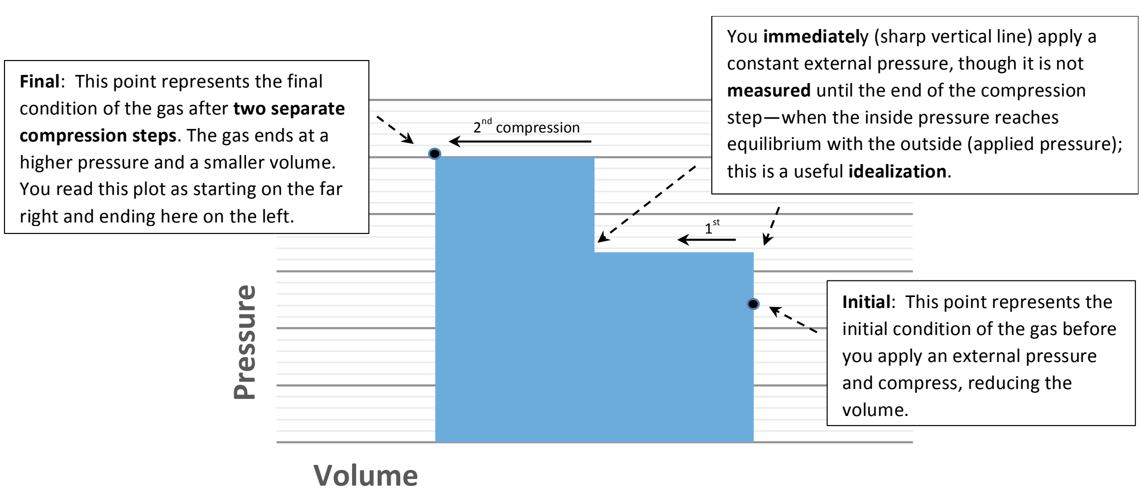
Goal: The goal of this activity is to **determine the amount of work done to compress an ideal gas** in 1-step, 3-steps, 5-steps, and an infinite number of steps (a hypothetical). It is assumed that you have limited knowledge of thermodynamics (a field that branches across chemistry and physics) and you are not responsible for any of the chemistry or physics content beyond being prepared for class. The idea is to have a better sense of integration by applying your knowledge to data you physically collect in real time.

Background: Geometry is often useful in explaining and visualizing arithmetic. The product of two multiplied numbers, for example, can be easily visualized as the area of a rectangle. The diagrams to the right can be thought of in similar ways ($3 \cdot 4 = 12$).

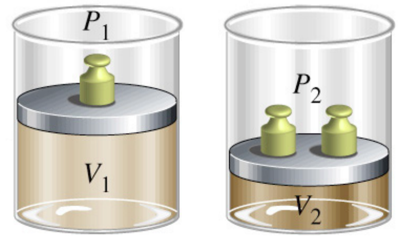


Let's consider another example that involves the calculation of work. The work associated with a constant force (F) applied in a straight line that changes an object?s position (x) follows a simple equation: $w = F \cdot \Delta x$. To the right we see 3 Newtons of force constantly applied to move an object from $x = 0$ meters to $x = 4$ meters. The work done, as shown to the right, equals 12 Newton-meters.

In the lab you will be looking at the work associated with the compression of a gas. Assuming a constant pressure is applied, the work done to compress the gas is the product of the pressure applied (closely related to force) and the change in volume (closely related to a change in position): $w = -P_{ex} \cdot \Delta V$. (See footnote.[3]) The area diagramed in the plot below represents the work associated with a two-step compression. **Carefully study the plot** (there is a lot more going on here than just areas, and this diagram is the key to the entire lab) and think about how much work, in total, was required to compress the gas (**also pay close attention to all the notes**).



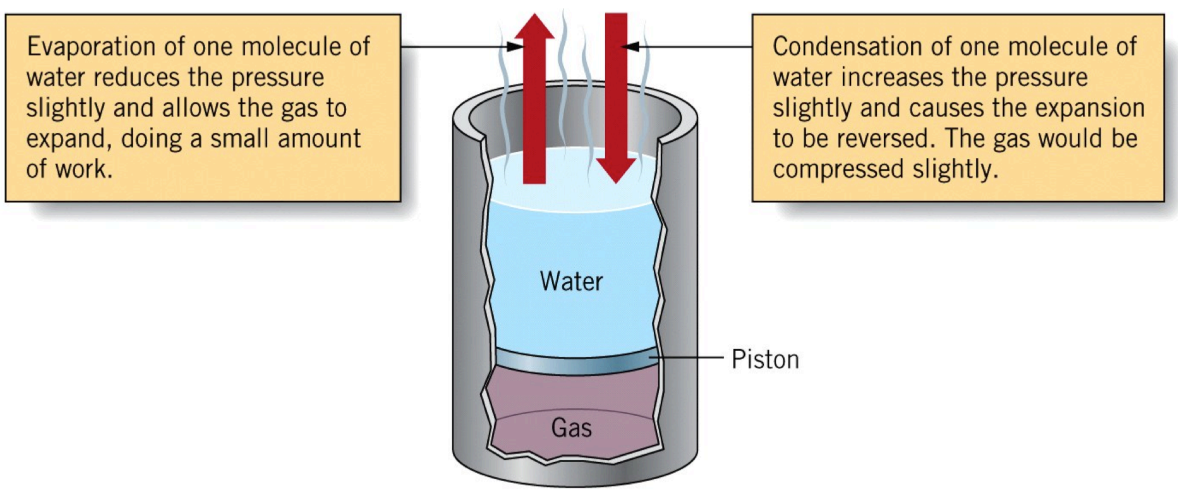
[3] The negative sign $-$ in $w = -P_{ex} \cdot \Delta V$ helps account for the negative change in volume ($V_f - V_i$) when you compress a gas. Therefore the work will remain positive; you do work to compress a gas.



It will be important to consider the pressure you apply to the gas with each step as an approximately constant value (though it may not feel this way as you push down). Ideally you would simply place a particular weight on the compression cylinder for each step. A ten-pound weight would compress the gas a certain volume and an additional ten pounds would further compress the gas. Visualizing adding weights helps understand the idealization where the **higher pressure is reached immediately** and then **remains constant throughout each step** in your plots of pressure vs. volume. This is an idealization and exaggerates the simplifications in $w = -P_{ex} \cdot \Delta V$ as applied to a real process.

Pre-Lab Questions (may well **require research**):

1. The work associated with a constant force (F) applied in a straight line that changes an object's position (x) along the direction of the force follows a simple equation: $w = F \cdot \Delta x$. If Pressure equals Force per unit Area (F/A), show that to compress a gas $w = -P_{ex} \cdot \Delta V$, where ΔV is a change in volume.
2. What is the ideal gas law? Label each variable and provide units.
3. Is air, the substance you will compress in the lab tomorrow, an ideal gas? Explain.
4. What is an integral? What is the difference between a definite and an indefinite integral?
5. Within the context of thermodynamics, what is a reversible process? How might the concept of a thermodynamically reversible process relate to calculus? See example below.



Procedure:

Total Cylinder Volume = 236ml +/− 2ml if the top position is 15.5cm

Total Cylinder Volume = 228ml +/− 2ml if the top position is 15.0cm

Volume per Centimeter = 15ml +/− 1ml

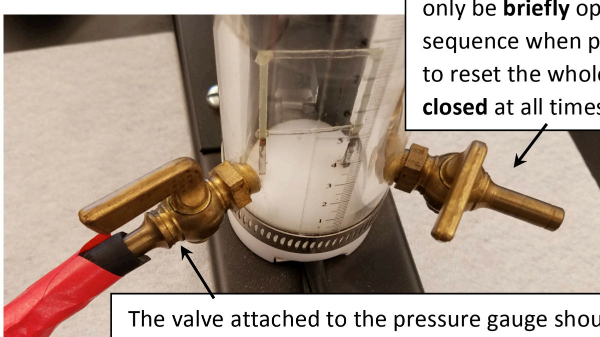
Add 15 psi to all pressure measurements: 0.0 psi on the guage = 15 psi

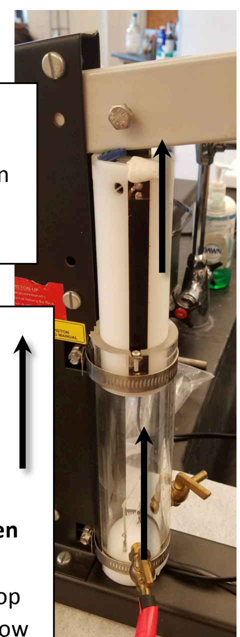
It is fine that your units of work will likely be psi·mL. 1 psi·mL = 0.00689 Joules.

(1) Spend a little time just "playing" with the gas compression chambers (but be gentle and careful!). Get a sense of how they work. Make sure you can open the turn values and reset the system pressure and position. Notice that the pressure gauge is reset with a push-button and will "stick" at the highest pressure recorded during a fast compression/relaxation. Be sure to check the total volume of your specific cylinder using the data in the procedure box at the bottom of the previous page!

(2) Press "collect" in Logger Pro and then "stop" so that you can see how to track temperature changes (especially max temperatures) for individual compression steps throughout the day.

(3) Spend a little time working with the valve system.





One-Step Compression

(4) Ask a partner to completely compress the gas in a single, fast step. Record the maximum measured pressure (again this should be thought of as approximately a constant pressure applied throughout the step — can be thought of as one really heavy weight dropped on the compression chamber). Record the final position of the compression plunger and convert this to a final volume. Record the maximum temperature reached during the compression.

(5) On your graph paper draw a diagram analogous to the one in the background information, but for a 1-step compression. Make it large, and especially broad, so that you can later draw 3- and 5-step diagrams with similar scales. The x and y axis do not need to start at $(0, 0)$; and only the area of interest needs to be to scale.

(6) Calculate the area under your plot. What does this value represent?

(7) Integrate the function $dw = -P_{ex} \cdot \Delta V$ where P_{ex} is a constant (numerically equal to the maximum pressure you recorded after adding 15psi to the gauge reading). Then show how the result is evaluated over the volume range you recorded showing all the steps used in the the standard notation applied to definite integrals. In what way is applying the original function, $dw = -P_{ex} \cdot \Delta V$, to these procedural steps an awkward use of the differential notation "d" (as in dw and dV)? Why can we get away with it?

Three-Step Compression (read all of 8-10 **before** proceeding)

(8) Ask a partner to completely compress the gas in three separate steps, stopping after each step and holding the plunger steady. These do not have to be exactly equal steps, just do your best.

(9) Record the maximum measured pressure after the first step. Record the position of the compression plunger after the first step. Repeat these recordings for each subsequent step, so that you end with three sets of pressure and position data, and maximum temperatures.

(10) Convert all your position data to final volumes.

(11) Now draw a diagram on your graph paper describing the 3-step compression. Make sure the scale matches that of the 1-step diagram you drew earlier. You can always adjust and redraw these as needed.

(12) Calculate the work required to compress the gas in 3 steps.

Five-Step Compression

(13) Repeat steps 7-11, but for a 5-step compression.

Finishing Up

(14) Make sure you have finished all your graph paper plots and answered any questions included in the procedure below each plot before you leave class today.

Lab Questions (again, may **require research**):

(1) Draw a diagram representing a compression via an infinite number of steps. Roughly sketch this diagram, it does not need to be perfect. Just imagine how the plot will change as the number of steps increases a whole lot! Try to do this on your own before ?finding? the answer.

(2) What is an isothermal process? Think about the temperature readings during the various compressions. Which type of compression 1-step, 3-step, or 5-step was the closest to an isothermal process?

(3) If you completed an essentially infinite number of steps and waited a long time after each step, you could approximate an isothermal compression. Explain why this makes sense.

(4) One way to think about the diagram drawn in response to question #1 at the top of the page is to imagine that we apply a new constant pressure an infinite number of times (drop infinitesimally small weights on the compression chamber again and again). Fortunately, we can also imagine that the pressure changes according to the ideal gas law; $P_{external} = P_{internal}$ if the steps are infinitesimally small. Rewrite $w = -P_{ex} \cdot \Delta V$ in terms of differential notation "d" (as in dw and dV). Now, **using the ideal gas law**, rewrite the equation such that P is in terms of **only** volume and nRT (each of these latter values can be thought of as essentially constant).

(5) Determine the antiderivative of the final function derived in #5 above. Remember that nRT can all be thought of as constant under the slow, infinite step, compression.

(6) Show how the result in #5 above is evaluated over the volume range you used in the lab. The appropriate R value in $J/(\text{mole} \cdot K) = 8.314 J/(\text{mole} \cdot K)$. To compare this with your lab data recall that $1\text{psi} \cdot \text{mL} = 0.00689$ Joules. Room temperature ($T = 298$K). Show your work using the **standard notation** applied to **definite** integrals. $n = 0.00965$ **moles of gas if the top position of your cylinder is 15.5cm;** $n = 0.00932$ **moles of gas if the top position of your cylinder is 15.0cm**.

(7) Does the "+C" constant of integration serve a purpose in our lab? Explain.

(8) Your calculations indicate that you did a lot more work in the 1-step than in the 5-step, and certainly more than in the hypothetical, infinite step compression. **Calculate** the difference in work required to compress the air in 1-step versus an infinite number of steps. Comment on the fact that work is a "**path dependent**" function.

(9) The fact that energy is conserved and that the final energy of the system (the compressed gas) is the same regardless of whether it was compressed in 1-step or an infinite number of steps, leads to an interesting question: Where did all the extra work you did to compress the gas in 1-step go? Comment on that question.

What to turn in: (Do not type anything!)

(1) The answers to all the pre-lab questions.

(2) All the diagramed plots completed during the lab procedure (or redrawn as needed to clear any errors). These should be on graph paper.

(3) The answers to any questions posed in the procedure. These answers should be written out below the respective diagram plots with which they are associated, on the exact same graph paper pages.

(4) The answers to the final set of lab questions.

acceleration: The rate of change of velocity.

accumulate: To integrate a function, which becomes the rate of accumulation.

angle-addition identities: For any angles α and β, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

antiderivative: If f is the derivative of g, then g is called *an antiderivative of f*. For example, $g(x) = 2x\sqrt{x} + 5$ is an antiderivative of $f(x) = 3\sqrt{x}$, because $g' = f$.

antidifferentiation by parts: A common approach to *integration by parts*.

AP Questions: A few free-response questions are found at

arithmetic mean: The *arithmetic mean* of two numbers p and q is $\dfrac{1}{2}(p+q)$.

arithmetic sequence: A list in which each term is obtained by adding a constant amount to the preceding term.

asymptote: Two graphs are asymptotic if they become indistinguishable as the plotted points get further from the origin. Either graph is an asymptote for the other. If one of the graphs is a vertical line, then we call it a *vertical asymptote* and similarly a horizontal line is called a *horizontal asymptote*.

average relative rate of change over an interval: For a function $y = f(t)$, the average relative rate of change of f over the interval $a \le t \le b$ is $\dfrac{f(b)-f(a)}{f(a)(b-a)}$. See *average percent rate of change over an interval*.

average velocity: Average velocity is displacement divided by elapsed time; it is calculated during a time interval.

binomial coefficients: Numbers that appear when a binomial power $(x+y)^n$ is multiplied out. For example, $(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$, whose coefficients are 1, 5, 10, 10, 5, 1 — this is the fifth row of *Pascal's Triangle*. See *combination*.

center of curvature: Given points P and Q on a differentiable curve, let C be the intersection of the lines *normal* to the curve at P and Q. The limiting position of C as Q approaches P is the center of curvature of the curve at P.

Chain Rule: If a composite function is defined by $C(x) = f(g(x))$, its derivative is a product of derivatives, namely $C'(x) = f'(g(x))g'(x)$.

chord: A segment that joins two points on a curve.

components of velocity: See *velocity vector*.

composite, composition: A function that is obtained by applying two functions in succession. For example, $f(x) = (2x - 60)^3$ is a composite of $g(x) = 2x - 60$ and $h(x) = x^3$, because $f(x) = h(g(x))$. Another composite of g and h is $k(x) = g(h(x)) = 2x^3 - 60$. Notice also that f is a composite of $p(x) = 2x$ and $q(x) = (x - 60)^3$.

compound interest: When interest is left in an account (instead of being withdrawn), the additional money in the account itself earns interest.

concave up/down: See *concavity*. It is said that a function is concave up/down at point $(p, f(p))$ if it is concave up/down on some interval containing p, for $a < p < b$.

concavity: A graph $y = f(x)$ is *concave up* on an interval if f'' is positive on the interval. The graph is *concave down* on an interval if f'' is negative on the interval.

conic section: Any graph obtainable by slicing a cone with a cutting plane. This might be an ellipse, a parabola, a hyperbola, or some other special case.

constant function: A function that has only one value.

continuity: A function f is *continuous at a* if $f(a) = \lim_{x \to a} f(x)$. A function is called *continuous* if it is continuous at every point in its domain. For example, $f(x) = 1/x$ (which is undefined at $x = 0$) is continuous. If a function is continuous at every point in an interval, the function is said to be continuous *on* that interval.

converge (series): If the *partial sums* of an infinite *series* come arbitrarily close to a fixed value, the series is said to *converge* to that value.

converge (integral): An *improper integral* that has a finite value is said to *converge* to that value, which is defined using a limit of proper integrals.

cosecant: The reciprocal of the sine.

critical point: A number c in the domain of a function f is called *critical* if $f'(c) = 0$ or if $f'(c)$ is undefined.

cross-sections: A method of calculating the volume of a solid figure, which partitions the solid by a system of 2-dimensional slices that are all perpendicular to a fixed axis.

curvature: In an absolute sense, the rate at which the direction of a curve is changing, with respect to the distance traveled along it. For a circle, this is just the reciprocal of the radius. The sign of the curvature indicates on which side of the tangent vector the curve is found.

cusp: A sharp point on a graph that is non-differentiable.

cylindrical shells: A system of thin-walled, coaxial tubes that dissect a given solid of revolution; used to set up a definite integral for the volume of the solid.

decreasing: A function f is *decreasing* on an interval $a \le x \le b$ if $f(v) < f(u)$ holds whenever $a \le u < v \le b$ does. It is said that a function is decreasing at point $(p, f(p))$ if it is decreasing on some interval containing p, for $a < p < b$.

degree: See *polynomial degree*.

derivative: Given a function f, its derivative is another function f', whose values are defined by $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, which is the derivative of f at a.

derivative at a point: Given a function f, its *derivative at a* is the limiting value of the difference quotient $\dfrac{f(x) - f(a)}{x - a}$ as x approaches a.

difference quotient: The slope of a chord that joins two points $(a, f(a))$ and $(b, f(b))$ on a graph $y = f(x)$ is $\dfrac{f(b) - f(a)}{b - a}$, a quotient of two differences.

differentiable: A function that has derivatives at all the points in its domain.

differential equation: An equation that is expressed in terms of an unknown function and its derivative. A solution to a differential equation is a function.

differentials: Things like dx, dt, and dy. Called "ghosts of departed quantities" by George Berkeley (1685-1753), who was skeptical of Newton's approach to mathematics.

differentiation: The process of finding a derivative, perhaps by evaluating the limit of a difference quotient, perhaps by applying a formula such as the *Power Rule*.

discontinuous: A function f has a *discontinuity at a* if $f(a)$ is defined but does not equal $\lim_{x \to a} f(x)$; a function is *discontinuous* if it has one or more discontinuities.

displacement:The length of the shortest path between an initial and terminal point. The actual path traveled by a particle is irrelevant.

diverge means *does not converge*.

domain: The domain of a function consists of all the numbers for which the function returns a value. For example, the domain of a logarithm function consists of positive numbers only.

double-angle identities: Best-known are $\sin 2\theta \equiv 2\sin\theta\cos\theta$, $\cos 2\theta \equiv 2\cos^2\theta - 1$, and $\cos 2\theta \equiv 1 - 2\sin^2\theta$; special cases of the *angle-addition identities*.

double root: A solution to an equation that appears twice. In the example $(x-5)^2 = 0$, $x = 5$ is a double root. This is also referred to as a root with multiplicity 2.

doubling time: The amount of time it takes for a population to double in size. This is a particular way of measuring compounding interest.

e is approximately 2.71828. This irrational number frequently appears in scientific investigations. One of the many ways of defining it is $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

eccentricity: For curves defined by a focus and a directrix, this number determines the shape of the curve. It is the distance to the focus divided by the distance to the directrix, measured from any point on the curve. The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle (a special ellipse) is 0. The word is pronounced "eck-sen-trissity".

ellipse I: An ellipse is determined by a focal point, a directing line, and an eccentricity between 0 and 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity.

ellipse II: An ellipse has two focal points. The sum of the *focal radii* to any point on the ellipse is constant.

equal-crust property: Given a solid of revolution cut by two planes perpendicular to the axis of revolution, this rare property says that the surface area found between the planes is proportional to the separation between the planes.

error: A quantifiable measurement of the distance a value is from the expected value. When using a linear approximation, the error is the difference between the y-value the line approximates in comparison to the y-value of the function at the same x-value. Therefore, error = predicted value − actual value.

Euler's Method: Given a differential equation, a starting point, and a step size, this method provides an approximate numerical solution to the equation.

even function: A function whose graph has reflective symmetry in the y-axis. Such a function satisfies the identity $f(x) = f(-x)$ for all x. The name *even* comes from the fact that $f(x) = x^n$ is an even function whenever the exponent n is an even integer.

expected value: The average value of a variable whose values are randomly determined by a probability experiment (the number of aces when three dice are tossed, for example). The average is calculated by considering every possible outcome of the experiment. It may be expressed as $p_1v_1 + p_2v_2 + p_3v_3 + \cdots + p_nv_n$, in which each value v_k has been multiplied by the probability p_k that v_k will occur. The expected value need not be a value of the variable — the expected number of heads is 2.5 when five coins are tossed. See also *weighted average*.

exponential functions have the strict form $f(x) = b^x$, with a constant base and a *variable exponent*. It is also common practice to use this terminology to refer to functions of the form $f(x) = k + a \cdot b^x$, although most of them do not satisfy the rules of exponents.

Extended Power Rule: The derivative of $p(x) = [f(x)]^n$ is $p'(x) = n[f(x)]^{n-1}f'(x)$.

extreme point: either a *local minimum* or a *local maximum*. Also called an *extremum*.

Extreme-value Theorem: If $f(x)$ is continuous for $a \le x \le b$, then $f(x)$ attains a maximum and a minimum value. In other words, $m \le f(x) \le M$, where $m = f(p)$, $a \le p \le b$, $M = f(q)$, and $a \le q \le b$. Furthermore, p and q are critical values or endpoints for f.

factorial: The product of all positive integers less than or equal to n is called *n factorial*. The abbreviation $n!$ is generally used. For example, 5! is 120. In general, $n!$ is the number of permutations of n distinguishable objects.

Fibonacci sequence: A list of numbers, each of which is the sum of the two preceding.

focal radius: A segment that joins a point on a conic section to one of the focal points; also used to indicate the length of such a segment.

frustrum: There is no such word. See *frustum*.

frustum: When a cone is sliced by a cutting plane that is parallel to its base, one of the resulting pieces is another (similar) cone; the other piece is a *frustum*.

functional notation: For identification purposes, functions are given short names (usually just one to three letters long). If f is the name of a function, then $f(a)$ refers to the number that f assigns to the value a.

Fundamental Theorem of Calculus: In a certain sense, differentiation and integration are inverse procedures.

geometric mean: The *geometric mean* of two positive numbers p and q is $\sqrt{pq}$.

geometric sequence: A list in which each term is obtained by applying a constant multiplier to the preceding term.

global maximum: Given a function f, this may or may not exist. It is the value $f(c)$ that satisfies $f(x) \le f(c)$ for all x in the domain of f.

global minimum: Given a function f, this may or may not exist. It is the value $f(c)$ that satisfies $f(c) \le f(x)$ for all x in the domain of f.

Greek letters: Apparently unavoidable in reading and writing mathematics! Some that are found in this book are α (alpha

half-life: When a quantity is described by a decreasing exponential function of t, this is the time needed for half of the current amount to disappear.

hole: The graph of $y = f(x)$ is said to have a hole at $x = a$ when $f(a)$ is undefined and $\lim_{x \to a} f(x) = c$, for some constant c.

horizontal asymptote: See *asymptote*.

Hypatia: Hypatia (c. 355-415) was a mathematician, astronomer, and philosopher who lived in Alexandria, Egypt. She is the first female mathematician whose life and work are reasonably well documented.

hyperbola I: A hyperbola has two focal points, and the difference between the *focal radii* drawn to any point on the hyperbola is constant.

hyperbola II: A hyperbola is determined by a focal point, a directing line, and an eccentricity greater than 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity.

identity: An equation (sometimes written using $\equiv$) that is true no matter what values are assigned to the variables that appear in it. One example is $(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$, and another is $\sin x \equiv \sin(180 - x)$.

implicit differentiation: Applying a differentiation operator to an identity that has not yet been solved for a dependent variable in terms of its independent variable.

implicitly defined function: Equations such as $x^2 + y^2 = 1$ do not express y explicitly in terms of x. As the examples $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ illustrate, there in fact could be several values of y that correspond to a given value of x. These functions are said to be *implicitly defined* by the equation $x^2 + y^2 = 1$.

improper fraction: Not a *proper fraction*.

improper integral: $\displaystyle\int_a^b f(x)\,dx$ is *improper* when a or b is infinite or when the integrand, $f(x)$, is not bounded on $[a, b]$ or is undefined for one or more x in $[a, b]$.

increasing: A function f is *increasing* on an interval $a \le x \le b$ if $f(u) < f(v)$ holds whenever $a \le u < v \le b$ does. It is said that a function is increasing at point $(p, f(p))$ if it is increasing on some interval containing p, for $a < p < b$.

indeterminate form: This is an ambiguous limit expression, whose actual value can be deduced only by looking at the given example. The five most common types are: $\dfrac{0}{0}$, examples of which are $\displaystyle\lim_{t\to 0} \frac{\sin t}{t}$ and $\displaystyle\lim_{h\to 0} \frac{2^h - 1}{h}$ 1^∞, examples of which are $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ and $\displaystyle\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$ $\dfrac{\infty}{\infty}$, examples of which are $\displaystyle\lim_{x\to\infty} \frac{2x+1}{3x+5}$ and $\displaystyle\lim_{x\to 0} \frac{\log_2 x}{\log_3 x}$ $0 \cdot \infty$, examples of which are $\displaystyle\lim_{x\to 0} x \ln x$ and $\displaystyle\lim_{x\to\pi/2} \left(x - \frac{1}{2}\pi\right) \tan x$ $\infty - \infty$, examples being $\displaystyle\lim_{x\to\infty} \sqrt{x^2 + 4x} - x$ and $\displaystyle\lim_{x\to\pi/2} \sec x \tan x - \sec^2 x$ The preceding limit examples all have different values.

infinite series: To find the sum of one of these, you must look at the limit of its partial sums. If the limit exists, the series *converges*; otherwise, it *diverges*.

inflection point: A point on a graph $y = f(x)$ where f'' changes sign.

instantaneous percent rate of change: For a function $y = f(t)$, the instantaneous percent rate change of f at $t = a$ is $\dfrac{f'(a)}{f(a)}$. This is also called the *instantaneous relative rate of change*.

instantaneous relative rate of change: See *instantaneous percent rate of change*.

instantaneous velocity: Instantaneous velocity is unmeasurable, and must therefore be calculated as a limiting value of average velocities, as the time interval diminishes to zero.

integral: The precise answer to an accumulation problem. A limit of *Riemann sums*.

integrand: A function that is integrated.

integration by parts: An application of the product rule that allows one to conclude that two definite integrals have the same numerical value. See examples below.

integration by substitution: Replacing the independent variable in a definite integral by some function of a new independent variable and adjusting the bounds appropriately. The integral that results has the same numerical value as the original integral.

intermediate-form: A limit expression is said to be in this form if the limit expression cannot be evaluated because it results in an undefined expression.

intermediate-value property: A function f has this property if, for any k between $f(a)$ and $f(b)$, there is a number p between a and b, for which $k = f(p)$. For example, f has this property if it is *continuous* on the interval $a \le x \le b$.

interval notation: A system of shorthand, in which "x is in $[a,b]$" or "$x \in [a, b]$" means "$a \le x \le b$" and "θ is in $(0, 2\pi)$" or "$\theta \in (0, 2\pi)$" means $0 < \theta < 2\pi$.

inverse function: Any function f processes input values to obtain output values. A function that undoes what f does is said to be *inverse* to f, and often denoted f^{-1}. In other words, $f^{-1}(b) = a$ must hold whenever $f(a) = b$ does. For some functions ($f(x) = x^2$, for example), it is necessary to restrict the domain in order to define an inverse.

isocline: A curve, all of whose points are assigned the same slope by a differential equation.

Law of Cosines: This theorem can be expressed in the SAS form $c^2 = a^2 + b^2 - 2ab\cos C$ or in the equivalent SSS form $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Leibniz notation: A method of naming a derived function by making reference to the variables used to define the function. For example, the volume V of a sphere is a function of the radius r. The derivative of this function can be denoted $\dfrac{dV}{dr}$ instead of V'. The philosopher Gottfried Wilhelm Leibniz (1646-1716) is given credit for inventing calculus (along with his contemporary, Isaac Newton).

l'Hôpital's Rule: A method for dealing with indeterminate forms: If f and g are differentiable, and $f(a) = 0 = g(a)$, then $\lim\limits_{t \to a} \dfrac{f(t)}{g(t)}$ equals $\lim\limits_{t \to a} \dfrac{f'(t)}{g'(t)}$, provided that the latter limit exists. The Marquis de l'Hôpital (1661-1704) wrote the first textbook on calculus.

limit: A number that the terms of a sequence (or the values of a function) get arbitrarily close to.

limiting value of a sequence: Let x_1, x_2, ... be an infinite sequence of real numbers. The sequence is said to converge to L, the limiting value of that sequence, provided that: for every $p > 0$ there is an integer N (which depends on p), such that $|x_n - L| < p$ is true whenever $N < n$.

linear approximation: Given a point $(a, f(a))$, the line tangent to $f(x)$ at $(a, f(a))$ gives a linear approximation to $f(x)$ for x-values near a. That is, $f(x) \approx f'(a)(x-a) + f(a)$. Notice that this corresponds with a Taylor series based at a with $n = 1$.

ln: An abbreviation of *natural logarithm*, it means $\log_e$. It should be read "log" or "natural log".

local linearity: When zoomed in on a point of a graph of a function, the function is locally linear at that point if the graph looks like a line.

local maximum: Given a function f and a point c in its domain, $f(c)$ is a local maximum of f if there is a positive number d such that $f(x) \le f(c)$ for all x in the domain of f that satisfy $|x - c| < d$.

local minimum: Given a function f and a point c in its domain, $f(c)$ is a local minimum of f if there is a positive number d such that $f(c) \le f(x)$ for all x in the domain of f that satisfy $|x - c| < d$.

logarithm: The exponent needed to express a given positive number as a power of a given positive base. Using a base of 4, the logarithm of 64 is 3, because $64 = 4^3$.

logarithmic derivative: Dividing the derivative of a function by the function produces a relative rate of change.

logistic equation: A differential equation that describes population growth in situations where limited resources constrain the growth.

long-division algorithm: The process by which an *improper fraction* is converted to a mixed fraction. For example, the polynomial division scheme shown at right was used to convert the improper fraction $\dfrac{2x^2+3}{x-2}$ into the equivalent mixed form $2x+4+\dfrac{11}{x-2}$. The process is terminated because the remainder 11 cannot be divided by $x-2$. (In other words, $\dfrac{11}{x-2}$ is a *proper fraction.*)

$$\begin{array}{r} 2x+4 \\ x-2\,\overline{)\,2x^2+0x+3} \\ \underline{2x^2-4x} \\ 4x+3 \\ \underline{4x-8} \\ 11 \end{array}$$

magnitude: For a complex number $a+bi$, the *magnitude* $|a+bi|$ is $\sqrt{a^2+b^2}$.

Mean-Value Theorem: If the curve $y = f(x)$ is continuous for $a \le x \le b$, and differentiable for $a < x < b$, then the slope of the line through $(a, f(a))$ and $(b, f(b))$ equals $f'(c)$, where c is strictly between a and b.

megabucks: Slang for a million dollars, this has been used in the naming of lotteries.

mho: The basic unit of conductance, which is the reciprocal of resistance, which is measured in ohms. This was probably someone's idea of a joke (according to Wikipedia, Lord Kelvin in 1883). The units mho and *siemens* are used interchangeably.

Mirzakhani: Maryam Mirzakhani (1977 - 2017) was an Iranian mathematician and professor at Stanford University. In 2014 she won the most prestigious award in mathematics, the Fields Medal, making her both the first woman and the first Iranian to be honored with this award.

mixed expression: The sum of a polynomial and a proper fraction, e.g. $2x-3+\dfrac{5x}{x^2+4}$.

moment: Quantifies the effect of a force that is magnified by applying it to a lever. Multiply the length of the lever by the magnitude of the force.

monotonic: For $x_1 < x_2$, a function f is said to be monotonic if either $f(x_1) \le f(x_2)$ for all x_i or $f(x_1) \ge f(x_2)$ for all x_i. Similarly, f is *strictly monotonic* if either $f(x_1) > f(x_2)$ for all x_i or $f(x_1) < f(x_2)$ for all x_i.

natural logarithm: The exponent needed to express a given positive number as a power of e.

Newton's Law of Cooling is described by exponential equations $D = D_0 b^t$, in which t repesents time, D is the difference between the temperature of the cooling object and the surrounding temperature, D_0 is the initial temperature difference, and b is a positive constant that incorporates the rate of cooling. Isaac Newton (1642-1727) contributed deep, original ideas to physics and mathematics.

Newton's Method is a recursive process for solving equations of the form $f(x) = 0$.

nondifferentiable: A function is nondiffererentiable at a point if its graph does not have a tangent line at that point, or if the tangent line has no slope.

normal line: The line that is perpendicular to a tangent line at the point of tangency.

nth derivative: The standard notation for the result of performing n successive differentiations of a function f is $f^{(n)}$. For example, $f^{(6)}$ means f''''''. It thus follows that $f^{(1)}$ means f' and $f^{(0)}$ means f.

oblate: Describes the shape of the solid that is produced by revolving an elliptical region around its minor axis. The Earth is an oblate ellipsoid. See also *prolate*.

odd function: A function whose graph has half-turn symmetry at the origin. Such a function satisfies the identity $f(-x) = -f(x)$ for all x. The name *odd* comes from the fact that $f(x) = x^n$ is an odd function whenever the exponent n is an odd integer.

one-sided limit: Just what the name says.

operator notation: A method of naming a derivative by means of a prefix, usually D, as in $D\cos x = -\sin x$, or $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$, or $D_x(u^x) = u^x(\ln u)D_x u$.

oval: A differentiable curve that is closed, simple (does not intersect itself), convex (no line intersects it more than twice), and that has two perpendicular axes of symmetry of different lengths.

Pandrosion: Pandrosion (c. 300-360) was a mathematician who flourished in the first half of the 4th century in Alexandria, Egypt. Although there was some confusion and disagreement over Pandrosion's sex, many current scholars believe she was female, and, thus, an even earlier female mathematician than Hypatia.

Pappus's Theorem: To find the volume of a solid obtained by revolving a planar region $\mathcal{R}$ around an axis in the same plane, simply multiply the area of $\mathcal{R}$ by the circumference of the circle generated by the centroid of $\mathcal{R}$.

parabola: This curve consists of all the points that are equidistant from a given point (the *focus*) and a given line (the *directrix*).

partial fractions: Converting a proper fraction with a complicated denominator into a sum of fractions with simpler denominators, as in $\dfrac{3x+2}{x^2+x} = \dfrac{2}{x} + \dfrac{1}{x+1}$.

partial sum: Given an infinite series $x_0+x_1+x_2+\cdots$, the finite series $x_0+x_1+x_2+\cdots+x_n$ is called the n^{th} *partial sum*.

Pascal's triangle: The entries in the n^{th} row of this array appear as coefficients in the expanded binomial $(a+b)^n$. The r^{th} entry in the n^{th} row is ${}_nC_r$, the number of ways to choose r things from n things. Each entry in Pascal's Triangle is the sum of the two entries above it. Blaise Pascal (1623-1662) made original contributions to geometry and the theory of probability. See *binomial coefficient* and *combination*.

percent error: $\dfrac{\text{predicted value - actual value}}{\text{actual value}} \cdot 100\%$.

period: A function f has positive number p as a period if $f(x+p) = f(x)$ holds for all x.

permutation: An arrangement of objects. There are ${}_nP_r = n \cdot (n-1) \cdots (n+1-r)$ ways to arrange r objects that are selected from a pool of n distinguishable objects.

piecewise-defined function: A function can be defined by different rules on different intervals of its domain. For example, $|x|$ equals x when $0 \le x$, and $|x|$ equals $-x$ when $x < 0$.

point-slope form: One way to write a linear equation with slope m that contains the point (h, k) is $y = m(x-h) + k$.

polynomial: A sum of terms, each being the product of a numerical coefficient and a nonnegative integer power of a variable, for examples $1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5$ and $2x^3 - 11x$.

polynomial degree: The degree of a polynomial is its largest exponent. For example, the degree of $p(x) = 2x^5 - 11x^3 + 6x^2 - 9x - 87$ is 5, and the degree of the constant polynomial $q(x) = 7$ is 0.

polynomial division: See *long division*.

Power Rule: The derivative of $p(x) = x^n$ is $p'(x) = nx^{n-1}$.

Product Rule: The derivative of $p(x) = f(x)g(x)$ is $p'(x) = f(x)g'(x) + g(x)f'(x)$.

prolate: Describes the shape of a solid that is produced by revolving an elliptical region around its major axis.

proper fraction: The degree of the numerator is less than the degree of the denominator, as in $\dfrac{5x-1}{x^2+4}$. Improper fractions can be converted by *long division* to *mixed expressions*.

proportional-crust property: Given a solid of revolution cut by two planes perpendicular to the axis of revolution, this rare property says that the surface area found between the planes is proportional to the volume found between the planes.

quadratic formula: The solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\ .$$

quadratic function: A polynomial function of the second degree.

quartic function: A polynomial function of the fourth degree.

Quotient Rule: The derivative of $p(x) = \dfrac{f(x)}{g(x)}$ is $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

radius of convergence: A *power series* $\Sigma c_n(x-a)^n$ converges for all x-values in an interval $a - r < x < a + r$ centered at a. The largest such r is the radius of convergence. It can be 0 or ∞.

radius of curvature: Given a point P on a differentiable curve, this is the distance from P to the *center of curvature* for that point.

random walk: A sequence of points, each of which is obtained recursively and randomly from the preceding point.

range: The range of a function consists of all possible values the function can return. For example, the range of the sine function is the interval $-1 \le y \le 1$.

Ratio Test: Provides a sufficient condition for the absolute convergence of $\sum a_n$.

real axis: See *complex-number plane.*

recursion: This is a method of describing a sequence, whereby each term is defined by referring to previous terms. Two examples of recursion are $x_n = 1.007x_{n-1} - 87.17$ $x_n = (1 + x_{n-1})\,/x_{n-2}$ To complete such a definition, initial values must be provided.

reduction formula: Recursively generates a sequence of integrals or antiderivatives.

relative maximum means the same thing as *local maximum.*

relative minimum means the same thing as *local minimum.*

Riemann sum: This has the form $f(x_1)w_1 + f(x_2)w_2 + f(x_3)w_3 + \cdots + f(x_n)w_n$. It is an approximation to the integral $\int_a^b f(x)\,dx$. The interval of integration $a \le x \le b$ is divided into subintervals I_1, I_2, I_3, $\ldots$, and I_n, whose lengths are w_1, w_2, w_3, $\ldots$, and w_n, respectively. For each subinterval I_k, the value $f(x_k)$ is calculated using a value x_k from I_k. Riemann (1826-1866) applied calculus to geometry in original ways.

Rolle's Theorem: If f is a differentiable function, and $f(a) = 0 = f(b)$, then $f'(c) = 0$ for at least one c between a and b. Michel Rolle (1652-1719) described the emerging calculus as a collection of ingenious fallacies.

root: Another name for *zero*.

Rugby: One of the oldest boarding schools in England, it is probably best known for a game that originated there, and for the clothing worn by the players of that game.

Sandwich Theorem: See *Squeeze Theorem*.

secant: The reciprocal of the cosine.

secant line: A line that intersects a (nonlinear) graph in two places.

separable: A differential equation that can be written in the form $f(y)\dfrac{dy}{dx} = g(x)$.

sequence: A list, typically generated according to a pattern, which can be described *explicitly*, as in $u_n = 5280(1.02)^n$, or else *recursively*, as in $u_n = 3.46u_{n-1}(1 - u_{n-1})$, $u_0 = 0.331$. In either case, it is understood that n stands for a nonnegative integer.

series: The *sum* of a sequence.

shells: A method of calculating the volume of a *solid of revolution*, which partitions the solid by a system of nested cylindrical shells (or sleeves) of varying heights and radii.

siemens: see *mho*.

sigma notation: A concise way of describing a *series*. For examples, the expression $\displaystyle\sum_{n=0}^{24} r^n$ stands for the sum $1 + r + r^2 + \cdots + r^{24}$, and the expression $\displaystyle\sum_{n=5}^{17} \frac{n}{24}$ stands for $\dfrac{5}{24} + \dfrac{6}{24} + \dfrac{7}{24} + \cdots + \dfrac{17}{24}$. The sigma is the Greek letter S.

signum function: This is defined for all nonzero values of x by $\operatorname{sgn}(x) = \dfrac{x}{|x|}$. This is also referred to as the *sign function.* Signum is the Latin word for sign.

simple harmonic motion: A sinusoidal function of time that models the movement of some physical objects, such as weights suspended from springs.

slope field: To visualize the solution curves for a differential equation $\dfrac{dy}{dx} = F(x, y)$, plot several short segments to represent the slopes assigned to each point in the xy-plane.

slope of a curve at a point: The slope of the tangent line at that point.

smooth curve: A curve with a continuous derivative.

solid of revolution: A 3-dimensional object that is defined by a region $\mathcal{R}$ and a line λ that lie in the same plane: the solid is the union of all circles whose centers are on λ, whose planes are perpendicular to λ, and that intersect $\mathcal{R}$.

speed: The magnitude of *velocity.* For a parametric curve $(x, y) = (f(t), g(t))$, speed is expressed by the formula $\sqrt{(x')^2 + (y')^2}$, which is sometimes denoted $\dfrac{ds}{dt}$. [120] Notice that that speed is *not* the same as dy/dx.

Squeeze Theorem: If $f(x) \le g(x) \le h(x)$, for all x in an interval (a, b) that contains c (except possibly at c) and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$.

standard position: An angle in the xy-plane is said to be in *standard position* if its initial ray points in the positive x-direction. Angles that open in the counterclockwise direction are *positive*; angles that open in the clockwise direction are *negative.*

step function: A piecewise constant function.

strictly monontonic: See monotonic.

surface of revolution: A 2-dimensional object that is defined by an arc $\mathcal{A}$ and a line λ that lie in the same plane: the surface is the union of all circles whose centers are on λ, whose planes are perpendicular to λ, and that intersect $\mathcal{A}$.

tail: Given an infinite series $a_1 + a_2 + a_3 + \cdots$, a *tail* is the infinite series $a_m + a_{m+1} + \cdots$ that results by removing the partial sum $a_1 + a_2 + \cdots + a_{m-1}$. For the given series to be convergent, the sum of this tail must become arbitrarily small when m is made suitably large; the converse is also true.

tangent line: A line is tangent to a curve at a point P if the line and the curve become indistinguishable when arbitrarily small neighborhoods of P are examined.

term-by-term differentiation: The derivative of a sum of functions is the sum of the derivatives of the functions.

Torricelli's Law: When liquid drains from an open container through a hole in the bottom, the speed of the droplets leaving the hole equals the speed that droplets would have if they fell from the liquid surface to the hole. Evangelista Torricelli (1608-1647) was the first to consider the graphs of logarithmic functions.

torus: A surface that models an inner tube, or the boundary of a doughnut.

trapezoidal method: A method of numerical integration that approximates the integrand by a piecewise-linear function.

triangle inequality: The inequality $PQ \le PR + RQ$ says that any side of any triangle is at most equal to the sum of the other two sides.

u**-substitution**: See *integration by substitution*.

vertical asymptote: See *asymptote*.

velocity vector: The *velocity vector* of a differentiable curve $(x, y) = (f(t), g(t))$ is $\left[\dfrac{df}{dt}, \dfrac{dg}{dt}\right]$ or $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$, which is tangent to the curve. Its magnitude $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ is the speed. Its *components* are derivatives of the component functions.

volume by cross-sections: See *cross-sections*.

volume by shells: See *shells*.

Wallis product formula is $\dfrac{\pi}{2} = \dfrac{2}{1} \cdot \dfrac{2}{3} \cdot \dfrac{4}{3} \cdot \dfrac{4}{5} \cdot \dfrac{6}{5} \cdot \dfrac{6}{7} \cdots = \lim_{k\to\infty} \left(\dfrac{4^k k!\, k!}{(2k)!}\right)^2 \dfrac{1}{2k+1}$. It was published in 1655 by John Wallis (1616-1703), who made original contributions to calculus and geometry.

weighted average: A sum $p_1y_1 + p_2y_2 + p_3y_3 + \cdots + p_ny_n$ is called a *weighted average* of the numbers y_1, y_2, $y_3, \ldots, y_n$, provided that $p_1 + p_2 + p_3 + \cdots + p_n = 1$ and each *weight* p_k is nonnegative. If $p_k = \dfrac{1}{n}$ for every k, this average is called the *arithmetic mean*.

Zeno's paradox: In the 5^{th} century BCE, Zeno of Elea argued that motion is impossible, because the moving object cannot reach its destination without first attaining the halfway point. Motion becomes an infinite sequence of tasks, which apparently cannot be completed. To resolve the paradox, observe that the sequence of times needed to accomplish the sequence of tasks is convergent.

zero: A number that produces 0 as a functional value. For example, $\sqrt{2}$ is one of the zeros of the function $f(x) = x^2 - 2$. Notice that 1 is a zero of any logarithm function, because $\log 1$ is 0, and the sine and tangent functions both have 0 as a zero.