Exeter Math Institute

Hands-on Algebra
Part B

Hands-on Algebra I – Part B

Table of Contents

Lab		Page Numbers
#1	The Number Line	p. 2
#2	Guess and Check Part B	p. 3-5
#3	Five Easy Pieces	p. 6-7
#4	Balance Beam	p. 8-10
#5	Factoring Tiles	p. 11
#6	Completing the Square	p.12-13
#7	Irrational Numbers	p. 14-15
#8	Paper Folding	p. 16-17
#9	What's Wrong with this Picture?	p. 18
#10	The Jumping Game	p. 19-20
#11	Two Squares Problem	p. 21-22
#12	Pythagorean Theorem Proof	p. 23
#13	Painted Cube	p. 24
#14	Pick's Theorem	p. 25-26
#15	Population Growth	p. 27-28
#16	Towers of Hanoi	p. 29-30
#17	Geoboard Slope	p. 31-32
#18	Geoboard y = mx + b	p. 33-34

The Number Line

Materials needed: post-it notes. About 3" square size is good. A yard/meter stick and/or some other rigid item for measuring units. A roll of paper tape is optional.

Directions:

Draw as large a number line as possible on the board or alternately use a piece of paper tape to make a model of the number line. Mark off 10 equal units as large as possible. Do not put a scale up yet.

1. Mark the left-most point 0 and the others to the right 0.1, 0.2, 0.3 etc. Write each number in Set #1 on a post-it note and distribute Set #1, one to each of six participants. Have them go to the front and post their number on the number line as accurately as possible on the number line, particularly noting the relationship of their number to the others. When they are finished, ask if everyone agrees with the relative position of each number.

   [Set #1: 0.044, 0.404, 0.440, 0.444, 0.0404, 0.04]

2. Remove the post-it notes from the first set and make a new set of post-it notes for Set #2. Mark the number line from 3.6 to 3.7 in hundredths. Repeat step 2.

   [Set #2: 3.66, 3.666, 3 2/3, 3.667, 3.67, 3.606]

3. Change the units now so that the left-most point is 0 and the right is 1. Put each of the numbers in Set #3 on its own post-it note and repeat step 2. Have students do this without a calculator.

   [Set #3: 1/4 , 1/3 , 2/5, 1/2 , 5/8]

4. Renumber the number line from –5 to 5. Make up post-it notes for step 5 and a set saying "the opposite of …". Then have them post both numbers.

   [Set #4: 3.46, 2.99, -0.98, 3.01, 0]]

5. Mark the number line in tenths from –2.0 to –1.0 and post the following set of numbers.

   [Set #5: -1.1, -1.01, -1.001, -1.011, -1.110}

6. With the same scale as in step 5, write an X on a post-it note and place it on the number line between 1 and 2, preferably not in the center. Write Set #5 on post-it notes and distribute them to each of four participants. You may also need a ruler, or some other rigid way to represent the length of one unit. Have each participant with a post-it note use the unit length to place his/her note on the number line.

   [Set #6: $x+2$, $x-3$, $4+x$, $-3-x$, $2-x$]

7. With the same number line, label a point between –1 and –2 as X. Plot the following set:

   [Set #7: the opposite of X, X – 4, X + 4, – 4 +X, – 4 – X]

Guess and Check Part B

Solving word problems often takes more time and effort than a computational math problem. However, they often can be done by thinking about how you might check an answer if you finally found one. In solving the following problems, try to suspend any previous methods that you might have learned and follow the directions step-by-step.

1. The length of a certain rectangle exceeds its width by exactly 12 centimeters. The perimeter of the rectangle is 58 cm. What are its dimensions?

Although you may be able to solve this problem using a method of your own, try the following approach, which begins by guessing the width of the rectangle. Study the first row of the table below, which begins with a guess of 10-cm. for the width. Now make your own guess for the width and use it to fill in the next row of the table. Be sure to show the arithmetic in detail that you used to complete each entry. If this second guess was not correct, try again.

width	length	perimeter	desired perimeter	check?
10 cm.	10 + 12 = 22 cm.	2(10)+2(22) = 64	58	no

Even if you have guessed the answer, substitute a w in the first column and fill in the length and perimeter entries in terms of w.

Finally, set your expression for the perimeter equal to the desired perimeter. Solve this resulting equation and thus solve the given problem.

This approach to creating equations and solving problems is called the ***guess-and-check*** method.

2. An I-Pod is on sale for 25% off its original price. The sale price is $195. What was the original price?

   In the table below, one guess has been made, which was not the correct answer. Make a guess of your own and put it in column 1. Continue to fill in across the row with the appropriate values. If your guess does not yield the correct solution, use another row of the table and guess again. You may add more rows, if needed. Be sure to show your calculations in the table.

Original price(**guess**)	25% of original price	sale price	given sale price	equal?
240	.25(240)=60	240-60=180	195	No

   Finally, place a variable in the first column and continue a cross the row, performing the same operations with the variable as you did with numbers, as far as possible. Form an equation by setting your expression in the “sale price” column equal to the number in the “given sale price” column. Solve the resulting equation in the space below the table and thus solve the given problem.

3. Jess would like to cut a board that was 50 inches long, into two pieces, one of which is 16 inches longer than the other. How long is each piece?

   Again, begin with a guess. After several guesses, or after finding the answer, remember to use a variable for the smaller piece and then supply the appropriate expressions in the row of the table. Use this row to form an equation that would allow you to find the answer.

Smaller piece	Larger piece	sum of the two pieces	Given sum	Check?

4. At noon, you start out walking to a friend's house at 4 mph. At the same time, your friend starts biking towards you at 12 mph. If your friend's house is 8 miles away, how much time will elapse before you two meet?

   In the table below, put your own headings on the columns before making a guess. It is usually a good idea to label the first column with what you are trying to find. After putting the titles on the columns, make your guess, form your equation and solve it as in the previous problems. You may use fewer or more columns than are shown, if you wish.

5. A restaurant has 23 tables. Some of the tables seat 4 people and the rest seat 2 people. In all, 76 people can be seated at once. How many tables of each kind are there?

Five Easy Pieces

Materials needed: an 8 ½ x11 piece of paper and pair of scissors for each participant.

Directions:

1. Begin with a blank piece of paper. Take the upper left-hand corner, fold it across to the right edge, and down towards the bottom in such a way as to form the largest possible right triangle. Cut off the flap at the bottom and unfold the paper so that you have the largest square possible.

2. Fold the square in half to form a rectangle.

3. Without unfolding, fold the rectangle in half to form a square. Continue by folding the square in half to form another rectangle. Fold the rectangle in half to form a small square.

4. Unfold the paper back to the original square. Cut out 1/4 of the original square and label it "U". Set it aside.

5. Cut another 1/4 of the original square. Cut out 4 smaller squares and label each "E". Set these aside.

6. Cut another 1/4 of the original square. Cut across the diagonal to form two congruent triangles and label each triangle "O". Set these aside.

7. On the last 1/4 of the original square place a dot at the midpoint of each side (on the fold line). Cut along the diagonals from dot to dot to form 4 congruent triangles. Label each small triangle "A". Set these aside.

8. Label the remaining square "I".

9. Complete the following chart by comparing the pieces you have. The first is done for you.

Question	Numerical answer	Written expression
How many As does it take to cover an E?	2	2A = E
How many As does it take to cover an I?		
How many As does it take to cover an O?		
How many As does it take to cover an U?		
How many Es does it take to cover an U?		
How many Es does it take to cover an O?		
How many Es does it take to cover an I?		

10. Referring to your written expressions,

    (a) If 4A = I and 8A = U, show algebraically the number of Is needed to equal one U.

    (b) If 4A = I and I = 2E, show algebraically the number of As needed to equal one E.

11. If you could cut up piece I, would it cover piece O? Justify your answer without actually cutting the pieces.

12. Write down all the possible ways that you can make the square U.

13. Take one E, one I and one U. These three squares renamed themselves: Bill, Jill and Lill.
Bill said, “I am bigger than Lill.”
Jill said, “I am bigger than Bill.”
Who are we?

14Take one A, one E and one O. These three pieces renamed themselves Ali, Bet and Tim.
Ali said, “I am twice as large as Bet.”
Bet said, “I am the same shape as Tim.”
Who are we?

15. Take one A, one E, one O and one U.
These four pieces renamed themselves: Mary Larry, Harry and Cary.
Mary said, “I am a fourth as large as Harry.”
Larry said, “I am bigger than Harry.”
Who are we?

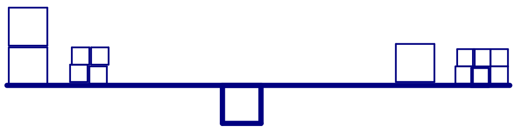
16. Take one A, one I, one O and one U. These four pieces re-named themselves Floe, Joe, Moe and Woe.
Joe said, “I am twice as large as Moe.”
Floe said, “I am the same shape as Woe.”
Woe said, “I am smaller than Moe.”
Who are we?

Balance Beam Lab

Materials needed: For each participant or pair of participants about 8 large blocks (dice in this lab) and about 15 smaller blocks (small wooden blocks in this lab) are needed. Candy such as caramel candy cubes (large blocks) and Starbursts (small blocks) can also be used. -

Directions:

1. Take a ruler and using two dice (one is behind the one shown in the picture below) as a fulcrum, balance the ruler as well as you can. On one side, place 2 large dice and 4 wooden blocks. On the other end, place 1 large die and 6 wooden blocks. Try to keep them towards the end with the dice on the outside. One arrangement is pictured below.

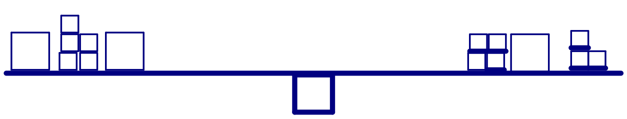


   Now try to remove blocks and dice so that you can determine how many blocks balance one die. Do this in such a way that the ruler is balanced at all times. (For example, you can take one block from the left side as long as you remove one block from the right side at the same time.)

   Record your result. _______ blocks equal one die.

2. Using a B for block and D for dice, write an equation that models the original grouping of blocks and dice as pictured above. Then show symbolically all the steps you took to determine your answer.

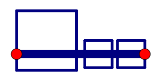
3. Clear your balance (ruler) and then form the situation pictured below using the large dice and blocks. Be sure that it will balance. Then try to remove blocks and dice, keeping the ruler balanced at all times. Your result should match the result obtained in problem 1.

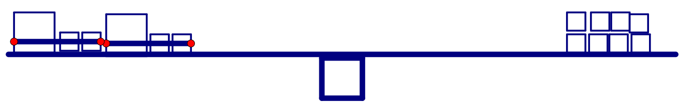


4. Using a B for block and D for dice, write an equation that models the original grouping of blocks and dice as pictured above. Then show symbolically all the steps you took to determine your answer.

5. Clear your balance (ruler) once again. On one side of the balance place 5 dice and 3 blocks and on the other side place 3 dice and 7 blocks. Try to stack the blocks and dice, and place them towards the ends of the ruler. Be sure that the ruler is balanced. Then try to remove dice and blocks as before. This time, you should reach the situation where 2 dice is balancing 4 blocks. Decide how to physically determine that 1die balances 2 blocks and also decide mathematically why you can do what you did.

6. Clear your balance. Take two blocks and one die and put a rubber band around them as pictured at the right. Do this again once more, making a total of two groupings. Place these on the left side of your ruler , stacking one on top of the other. Balance them by placing 8 blocks on the right side of the ruler as pictured.





The challenge now is to remove blocks and dice, as you did previously, to find the number of blocks that equal one die. However, before the blocks on the LHS can be removed from the ruler, what must be done first?

____

Complete that step and then proceed as before.

7. Problem number 6 can be modeled in symbols as follows: $\mathbf{2(}\boldsymbol{D}\mathbf{+2}\boldsymbol{B}\mathbf{)=8}\boldsymbol{B}$

If you wish to now show symbolically the steps taken in problem 6, what algebraic operation must be performed to remove the parentheses?

____

What physical step does that parallel in your work in problem 6?

____

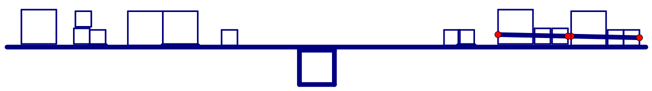
Perform this step and then continue to show the number of blocks that balance one die.

8. Returning to the set-up in problem 6, you may notice that there is a second way to approach the problem. What might be another first step?

____

Show this step symbolically by starting with $\mathbf{2(D + 2B) = 8B}$

9. Construct the situation pictured below on your balance and then proceed to remove blocks and dice to again show the number of blocks that balance one die.



10. Model the problem with variables and show symbolically how you arrived at your answer.

Factoring Trinomials With Tiles

Materials needed: Set of commercial tiles or hand-made tiles such as those on the next page.

Directions: Count out the following number of tiles for each group/pair of participants.

	Number needed
- small blue squares x by x called x^2-blocks	3
- large blue squares y by y called y^2-blocks	2
- small yellow 1 by 1 squares called 1's	10
- blue 1 by x rectangles called x-blocks	6
- blue 1 by y rectangles called y-blocks	5
- blue x by y rectangles called xy-blocks	3

Make a rectangle

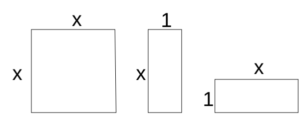
With the above resources you are going to try to make rectangles. For each set go through the following steps.

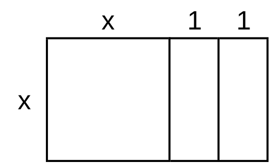
1. Write an expression for the sum of the areas of the resources. Combine like terms whenever possible.
2. Fit all the pieces together to form a rectangle.
3. Write the length and width of your new rectangle.
4. Write an expression for the area of your rectangle in the form (*length*)(*width*).
5. Write an equation that states that the sum of the areas of the individual resources (the expression you wrote in 1.) equals the area of your new rectangle (the expression you wrote in 4.)

example

Given the resources 1 x^2-block, 2 x-blocks:

1. The sum of the areas is $x^2 + x + x = x^2 + 2x$
2. Fit the pieces to form a rectangle (see lower diagram)
3. The length is $x + 2$, the width is x
4. An expression for the area is $x(x + 2)$
5. $x^2 + 2x = x(x + 2)$





you try these

1. 1 x^2-block, 3 x-blocks, 2 1's
2. 1 x^2-block, 2 x-blocks, 1 1 (yellow square)
3. 2 x^2-blocks, 4 x-blocks
4. 1 y^2-block, 4 y-blocks, 3 1's
5. 1 y^2-block, 4 y-blocks, 4 1's
6. 1 x^2-block, 1 y^2-block, 2 xy-blocks
7. 2 y^2-blocks, 5 y-blocks, 2 1's
8. 2 x^2-blocks, 1 y^2-blocks, 3 xy-block
9. 2 x^2-blocks, 2 x-blocks, 2 xy-blocks
10. 1 x^2-block, 1 y^2-block, 2 xy-blocks, 3 x- blocks, 3 y-blocks, 2 1's

11. Take one x^2-block and 6 x-blocks. Using all seven of these blocks each time, and any number of 1's, form as many different rectangles as you can. Write down the dimensions of each rectangle formed.

Completing the Square

Materials needed: Set of commercial tiles or hand-made tiles such as those described in the Factoring Trinomials with Tiles lab.
Directions: In this worksheet you are given some blocks and you have to decide how many yellow unit blocks you need to add to *make a square.*

1. (a) Take an x^2-block and 2 x-blocks. Show how you can make a square with these blocks if you add one yellow unit block.
   (b) What are the length and width of the square you built? length _________ width ________
   (c) Complete the following. The blank on the left side represents what you have to add. The blank on the right represents the length of the side of the square that you made.

$$x^2 + 2x + \_\_ = x^2 + 2x + (\_\_)^2 = (\_\_\_\_\_)^2$$

2. (a) Take an x^2-block and 4 x-blocks. How many yellow unit blocks must you add to make a square? _______
   (b) What are the length and width of the square you built? length _________ width ________
   (c) Complete the following: $x^2 + 4x + \_\_\_ = x^2 + 4x + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_)^2$

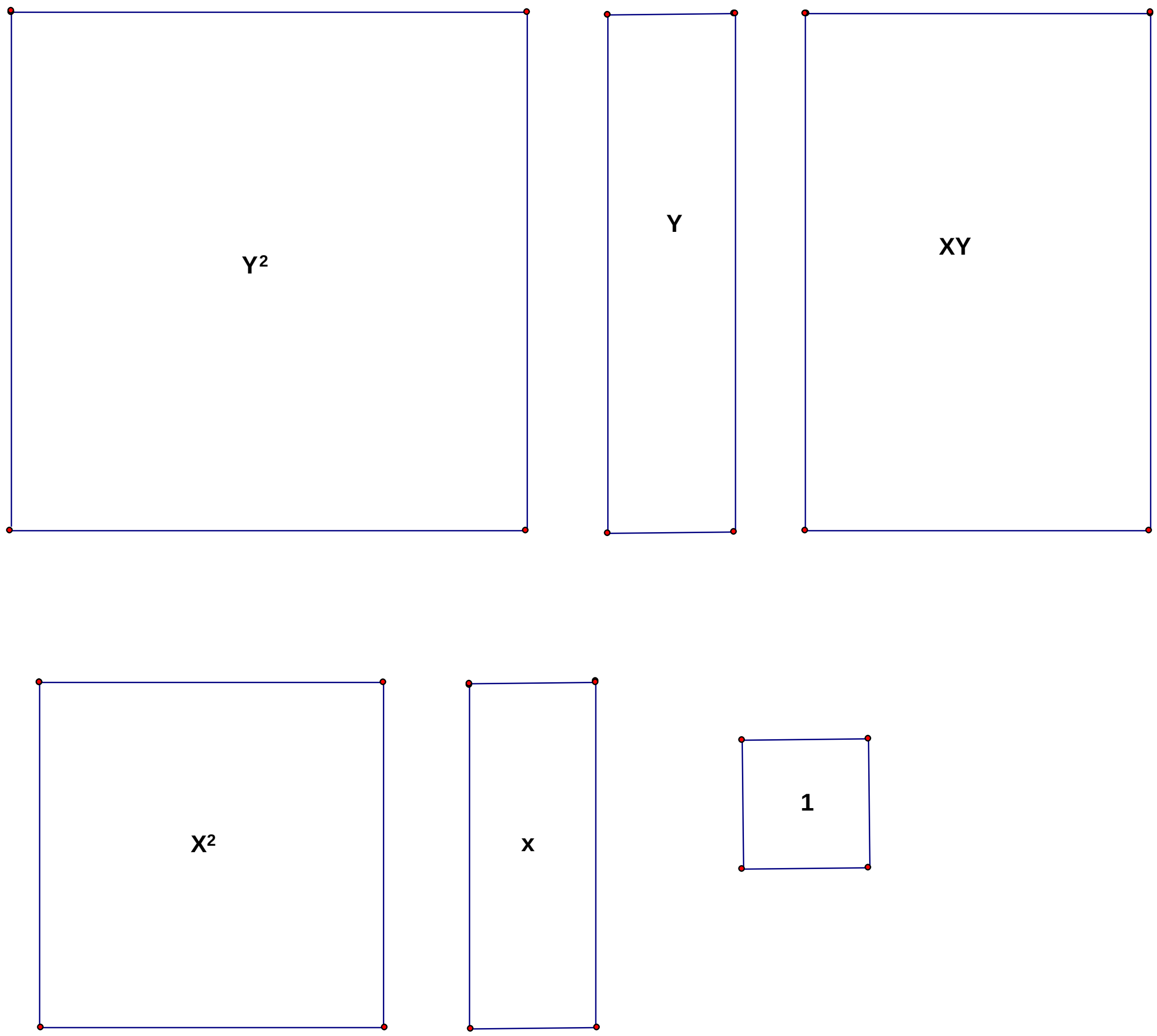
3. (a) Take an x^2-block and 6 x-blocks. How many yellow unit blocks must you add to make a square? ___________
   (b) What are the length and width of the square you built? length _________ width ________
   (c) Complete the following: $x^2 + 6x + \_\_\_ = x^2 + 6x + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_)^2$

4 (a) Suppose you take one x^2-block and 100 x-blocks. Can you predict how many unit blocks you would have to add to make a square? ______ How many? ___________
   (b) What would the length and width of your square be? length _________ width ________
   (c) Complete the following: $x^2 + 100x + \_\_\_ = x^2 + 100x + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_)^2$

5. (a) Suppose you take one x^2-block and *one* x-block. Imagine you could split the x -block into two blocks, each ½ by x. How much of a yellow block would you have to add to complete a square? Draw a diagram of this situation. Make sure you label all the dimensions.
   (b) What would the length and width of your square be?
   length _________ width ________
   (c)Complete the following:
   $x^2 + 1x + \_\_\_ = x^2 + 1x + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_)^2$

6. (a) Suppose you take one x^2-block and p x-blocks. Can you predict how many unit blocks you would have to add to complete the square?
   (b) What are the length and width of the square that you built? length _______ width______
   (c) Complete the following: $x^2 + px + \_\_\_ = x^2 + px + (\_\_\_)^2 = (\_\_\_\_\_\_\_\_)^2$

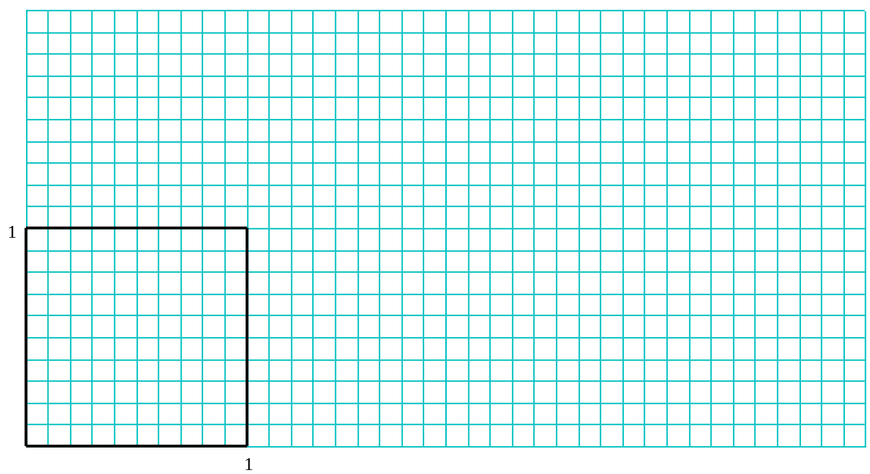
Template for factoring tile



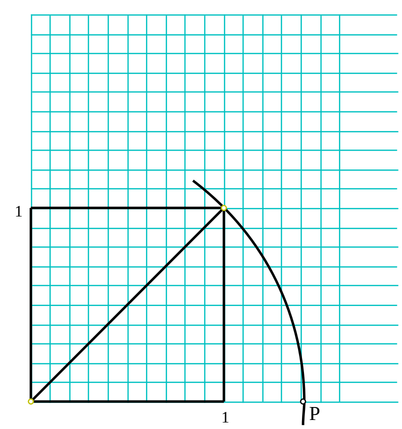
Irrational Number Worksheet

Materials needed: Graph paper, straight edge, compass

1. Using the straight edge, construct a square of unit length in the first quadrant using the points (0,0), (1,0), (0,1), and (1,1) as the vertices. In order for the square to be a good size, use 10 blocks on the graph paper to equal one unit length. Put the origin in the lower left hand corner of the graph paper with the longest side of the graph paper arranged horizontally.



2. Use the straight edge to construct the diagonal from (0,0) to (1,1). Use the Pythagorean Theorem to confirm the exact length of this diagonal is $\sqrt{2}$.

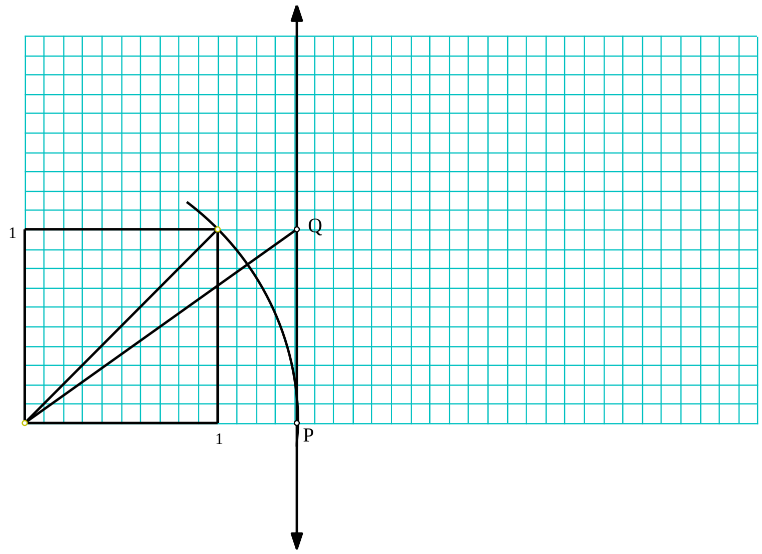


3. By placing the point of the compass at (0,0) and the pencil end of the compass on (1,1), construct an arc of a circle that intersects the *x*-axis at a point labeled P. What is the exact distance from the origin to point P?

4. Hopefully you answered the above question with $\sqrt{2}$. Using the squares on your graph paper, give an approximation to the nearest tenth for $\sqrt{2}$. Use your calculator to check to see if your approximation is correct.

5. The $\sqrt{2}$ is an example of an *irrational number*. Irrational numbers exist on the number line, but they cannot be expressed exactly as a decimal or a fraction. Even the number on your calculator is only a very accurate approximation for $\sqrt{2}$. *Rational numbers* are those numbers on the number line that can be expressed as decimals or fractions. Use the calculator approximation to obtain a decimal approximation for $\sqrt{2}$ .Between what two integers on the *x*-axis would you locate $\sqrt{2}$? If you divide your *x*-axis into tenths, between what two tenths would you locate $\sqrt{2}$? Now determine two rational numbers that are 0.001 apart, yet the $\sqrt{2}$ lies in between them. How about two rational numbers that are 0.00001 apart that have $\sqrt{2}$ in between them?

6. Using the straight edge, construct a vertical line through point P. Label where this vertical line intersects the horizontal line 1 unit up as point Q. Use the right triangle formed by the origin, and points P and Q to determine the exact length of the segment from the origin to point Q.



7. Hopefully, you calculated this length to be exactly $\sqrt{3}$. Now use the compass in a manner similar to what you did in the previous example to determine the location of $\sqrt{3}$ on the *x*-axis. Counting blocks on your graph paper, give an approximation for $\sqrt{3}$, and then check your answer using the calculator.

8. Use the calculator approximation of $\sqrt{3}$ to find two rational numbers on the *x*-axis line 0.01 apart, yet the $\sqrt{3}$ lies in between them. How about two rational numbers that are 0.000001 apart that have $\sqrt{3}$ in between them?

9. Use the straight edge and compass method along with the Pythagorean Theorem to locate $\sqrt{5}$ on the *x*-axis the same way you located $\sqrt{2}$ and $\sqrt{3}$.

A challenge: Locate $\sqrt{7}$ on the *x*-axis through a compass- straight edge construction.

Paper folding

Materials needed: 8 ½ x 11 paper, a few sheets of newspaper or larger paper to demonstrate maximum number of folds possible.

Directions:

1. How many times is it possible to fold a piece of paper? Take a piece of 8½ x 11 paper and fold it in half. You now have something that is two pages thick. Fold that piece in half again. How thick is it now? Continue folding and recording your results in the second column of the table at the right for 6 folds.

2. If one piece of paper is 0.003 inches thick, how thick would two pages be? Four pages? Record the thickness in inches in the third column of your table for all six folds.

number of folds	thickness in pages	thickness in inches
1	2	0.006
2	4	
3		
4		
5		
6		

3. It is virtually impossible possible to fold the paper 15 times. However, we could cut and stack the paper and obtain the same results. Looking back at step 1, we could have cut the paper in half and put one on top of the other. We could then have cut the pile in half and stacked it, obtaining a stack of four sheets. After doing this 13 more times for a total of 15, how many pieces of paper would be in the stack? At 0.003 inches per piece of paper, how high would the stack now be? Record these numbers in your table. Convert your answer to feet, showing your calculations.

4. Based on your work thus far, how thick in feet would you <u>estimate</u> that a stack formed by 25 cutting and stacking operations would be? Write your estimate below. Now compute the actual thickness of the stack formed by doing this operation 25 times and record your answer. Make use of your calculator to compute this answer.

   my estimate ________________ my answer from the calculator ____________________

5. Form a formula that gives the thickness *in feet* of the folded paper as a function of the number of folds and write it in the space below.

6. Mount Everest, the tallest mountain in the world is 29,028 feet high. About how many cutting and stacking operations would it take to get a stack at least this high? (Record your answer and also describe how you got it.)

7. The moon is 286,000 *miles* away. How many cutting and stacking operations would it take to form a stack that would reach the moon?

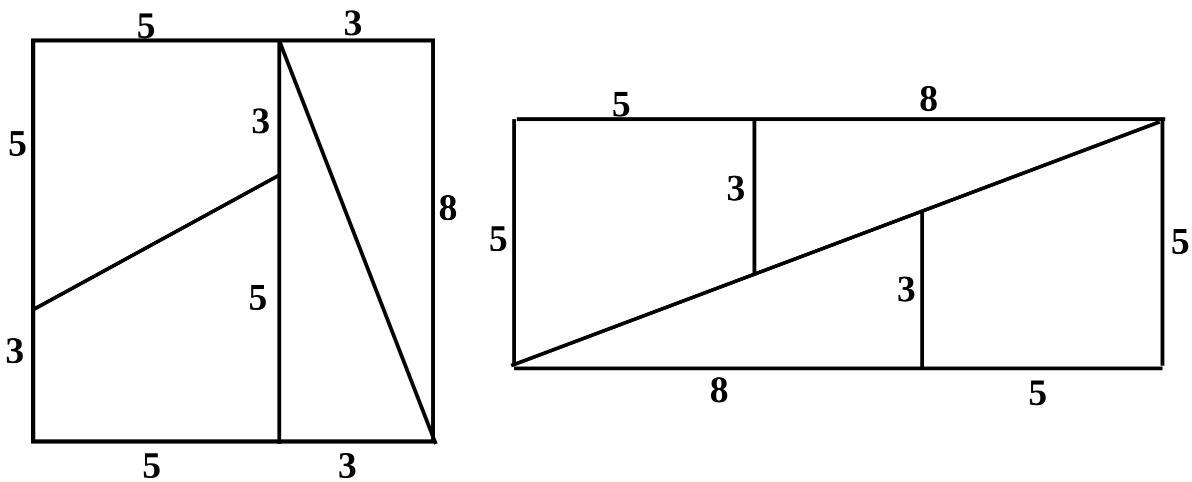
8. If we performed the cutting and stacking operation 24 times we would obtain a stack that is almost 0.8 miles high. (Verify this.)
   (a) If we would want to try to actually do this task, how many sheets of paper would be in the stack?

   (b) If each sheet in the stack was one square foot in area, what would be the dimensions of a large square piece of paper that could be used to accomplish this task?

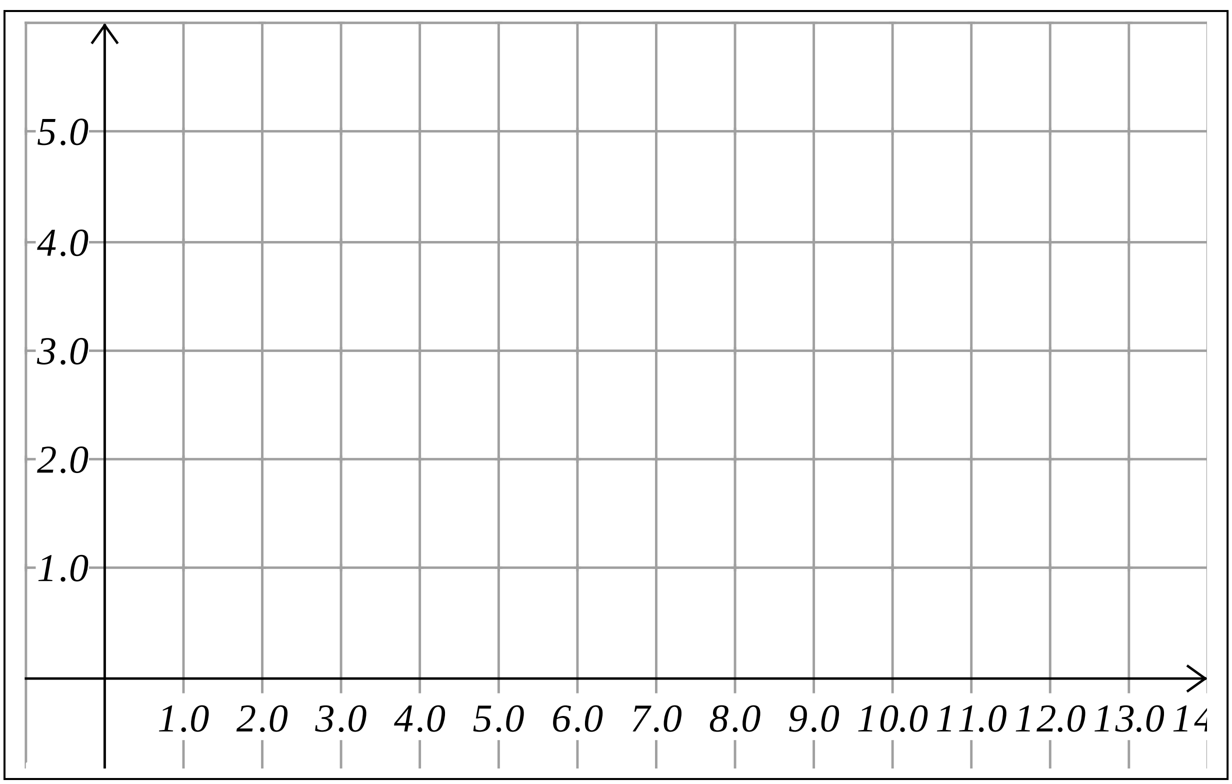
What Is Wrong With This Picture?

Directions:

Study the two pictures below. Is there anything wrong with what you see? Assume that the first figure is a square.



2. Draw the right-hand figure as accurately as possible on the graph below. Do the four pieces from the square fit perfectly in the rectangle? That is the area of the given square? ______? What is the area of the rectangle you formed? _____ Can you determine why they differ?



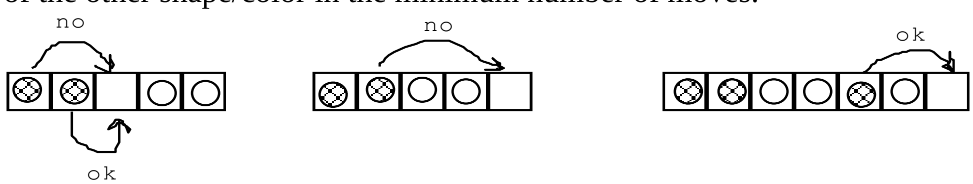
The Jumping Game

Materials needed: Two groups of different objects, four objects in each group. Colored disks work well. A sheet of paper on which to re-draw the figures in step 2 if these are not large enough to accommodate the disks.

The following game is based on these rules:

1. Moves can be made in only one direction. The objects cannot move backwards.
2. You may move into an adjacent unoccupied space.
3. You may jump a mover of the opposite shape/color, but you may only jump over one mover on each jump.

The object of each game is to completely switch the positions of the movers of one shape/color with the movers of the other shape/color in the minimum number of moves.



1. Place two different markers in the positions indicated on the game board at the right. What is the minimum number of moves necessary to completely switch their positions?

X		O

2. Now try the same thing with each of the following games and record the least number of moves needed in each game in the table provided. You may find it helpful to re-draw each game board on another piece of paper.

X	X		O	O

X	X	X		O	O	O

X	X	X	X		O	O	O	O

# of markers of one color	minimum # of moves
1	
2	
3	
4	

3. Is the pattern in the second column linear? ________ quadratic? ________ exponential? ________

Give a reason for your answer.

____

4. Determine an equation that describes the relationship between the two columns. ______________

It might be helpful to continue the pattern "backwards" to zero "# of markers of one color."

Your equation: _________________________

5. Return to the original problem and re-do the moves for each game. Record whether the move was a "slide" into the next space or a "jump" over one of the markers. For example, in the first game, the moves might be recorded as

S J S

What would be the pattern in the second game look like ? ____________________________

And the third? ____

The fourth ? ____

Fill in your values in the tale at the right: Remember that the two entries must total to be the same as the corresponding entry in your last table.

# of markers of one color	Number of jumps	Number of slides
1		
2		
3		
4		
n		

6. Let *n* represent the number of markers of one color. How could you represent the number of jumps in terms of *n*? _______

How could you represent the number of slides in terms of *n*? _______

What is the total number of moves in terms of *n*. ___________________ How does this compare with the formula obtained in step 4?

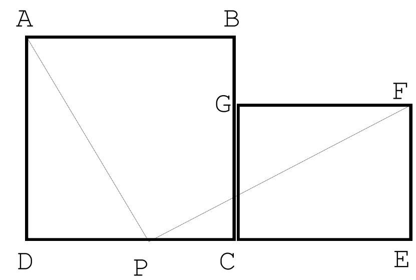
____

The Two-SquaresProblem

Materials needed: template for the two squares constructed on graph paper with a grid large enough so that the pieces are not too small when cut out.

Directions:

1. The two squares on the graph paper have sides of length 12 cm and 5 cm respectively. Point P is marked so that DP = 5 cm. and lines AP and PF are drawn. Cut out the two squares and then cut along AP and PF to form five separate pieces. Try to assemble the five pieces into a square.
   What is the length of the side of the new square? ___________________



2. We are assuming the figure we formed is a square. How do we know all the sides are equal?

   ____

   ____

   How do we know we have a right angle? _____________________________________

   ____

3. Will this work with another set of squares? On the remaining piece of graph paper, draw an 8 cm by 8 cm square in the corner, similar to ABCD on the one pictured on the next page. Next to it draw a 4 cm by 4 cm square CEFG. Locate point P 4 cm. from D and draw lines AP and FP. Place a letter on each of the five pieces for identification. Cut out the pieces and try to form a square.
   What is the length of the side of the new square? ___________________

4. With your partner, verify that we indeed have a square.

5. ABCD is a square with side 12 cm and CEFG is a square with side 9 cm. If you place them side-by-side, where would you locate the point P so that the two squares could be cut and formed into one large square?

   ____

   ____

   What is the length of the side of the new square? ___________________

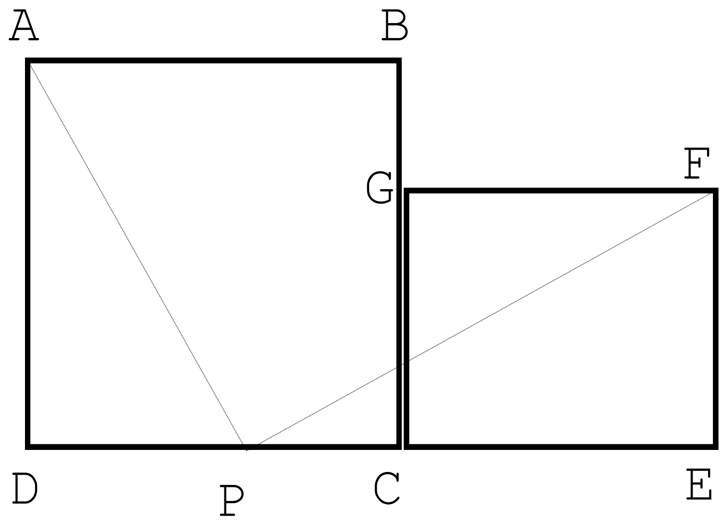
6. If two squares, each with a side of 10 cm. were placed side-by-side, how could you cut them so as to make it possible to rearrange them into one square?

____

7. On the picture below, label AD "***a***" and FE "***b***". Express the length of AP in terms of ***a*** and ***b***. Can this always be done?

____

8. Label PC and CE in terms of ***a*** and ***b***. How long is FP in terms of ***a*** and ***b***? ____________



9. In the space below is one version of the square you originally made out of the 12 cm and 5 cm squares. Assuming AD is now "**a**" and FE is now "**b**" as in steps 7 and 8, label FP and AP with the expressions you found in steps 7 and 8.

10. What is the area of this new square? __________

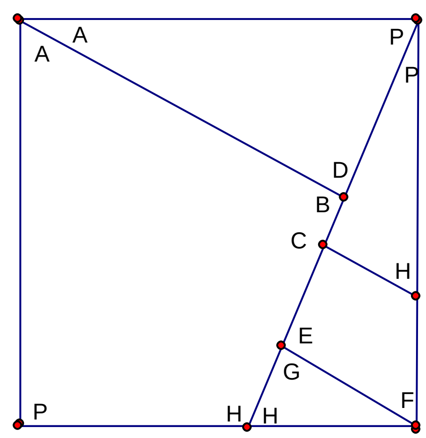
11. If we were to now label each side of this new square c, what would be its area in terms of c? _____

12. State the relationship between your answers in steps 10 and 11.

____

13. Complete this statement: The area of the new large square is equal to the sum of

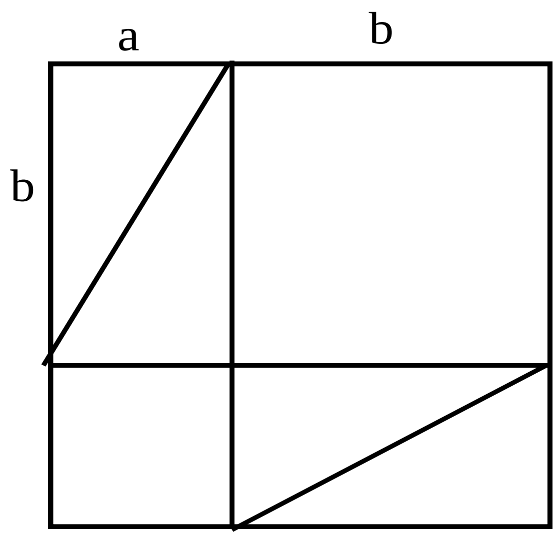
____

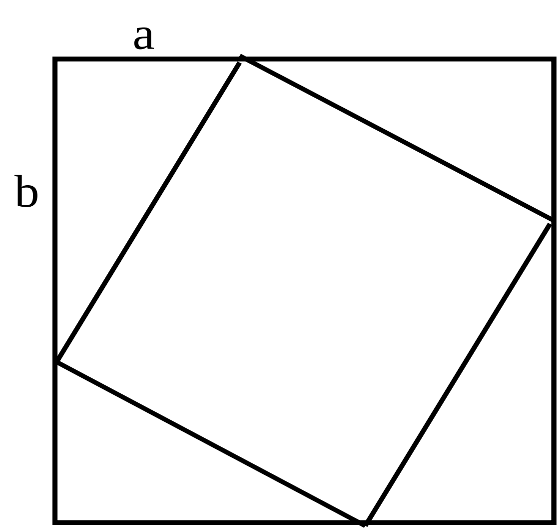


Pythagorean Proof Problem

The two-part diagram below, which shows two different dissections of the same square, was designed to help you *prove* the Pythagorean Theorem. In the drawing at the left, assume that the two interior intersecting lines are parallel to their respective sides.

1. Label all the lengths that you can with either an ***a*** or a ***b***.





2. What are the areas of the two interior squares in the left-hand square? Label appropriately.

3. Label the hypotenuse of each triangle in the left-hand square ***c***. Label the area of the square in the right-hand square in terms of ***c***.

4. Use a geometric argument, justify the Pythagorean Theorem.

___

___

___

5. Using an algebraic argument, justify the Pythagorean theorem.

The Painted Cube

Materials needed: 27 wooden cubes per group. This allows for the making of the first cube completely and the others partially.

Directions:

Imagine that you have 27 wooden cubes that you make into a large 3x3x3 cube. Then you paint all six faces of the outside of the large cube Exeter red. If you then take the cube apart into its 27 separate cubes, you will find that some cubes have more painted faces than others. Answer the following questions and enter your answers in the table below.

1. How many cubes are painted on exactly three faces?
2. How many cubes are painted on exactly two faces?
3. How many cubes are painted on exactly one face?
4. How many cubes have no paint?
5. Check that your answers to Questions 1 through 4 add up to 27.
6. Now suppose that you make a large 4x4x4 cube made with 64 small wooden cubes. Answer Questions 1 through 4 for this new situation. Enter your answers in the table below. Check that the number of cubes adds up to 64. Repeat this process for a large 5x5x5 cube made with 125 small wooden cubes.
7. By observing the data you have collected in the table and looking for patterns in each column, extend the table for a 6x6x6 cube.

Dimensions	Number of Cubes with				Total Number of Cubes
	Three Faces Painted	Two Faces Painted	One Face Painted	No Faces Painted	
3x3x3					
4x4x4					
5x5x5					
6x6x6					
NxNxN					

8. Suppose you have an NxNxN cube with n^3 small wooden cubes. Answer Questions 1 through 4 by writing formulas that describe the pattern in each of the four columns. What must be the sum of these four formulas? In the space below, show that the four formulas indeed add up to n^3

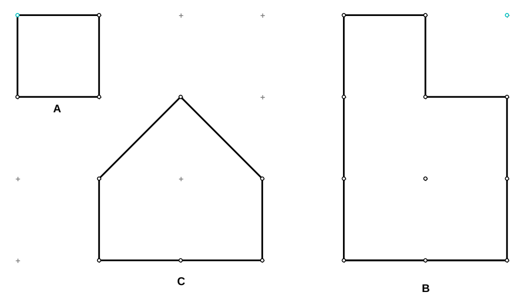
.

Pick’s Theorem

Materials needed: an 11-pin Geoboard for each participant along with some rubber bands.

Directions:

Examine the geoboard figures pictured at the right. Figure A has an area of 1 square unit and its perimeter has 4 pins. Figure B has an area of 5 square units. Its perimeter contains 10 pins, but it also encompasses 1 pin in its interior. Figure C also encloses one pin with 6 pins on its perimeter and has an area of 3 square units. Is there a relationship between the area of a polygon on a geoboard and the number of pins that the figure has on its perimeter and/or interior?



First Investigation:

area	number of pins
1	
2	
3	

1. Let’s look at figures that have no pins in their interiors. On your geoboard, form a square with area 1, then a rectangle of area 2. Record the number of pins on the perimeter of each in the table provided.

2. Form a polygon that has an area of 3 square units and contains no interior pins. Note that the figure can be done more than one way, as a rectangle, as an “L” shaped figure and as a triangle. Form all three and verify that the perimeter of each has the same number of pins. Record that number in the table.

3. Do the same for a convex polygon of area 4, with no interior pins, and record that number. Again, you should check several options to see that the number of pins in the perimeter is the same.

4. Do you see a pattern in your entries? ________

   Explain why the pattern is linear.

5. Let P represent the number of pins in the perimeter of the figure and A represent the area. Write an equation that relates the area to the number of pins in the perimeter. Solve the formula for A.

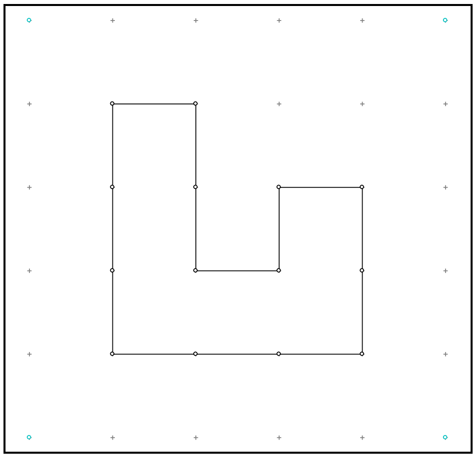
6. Test the formula on the figure at the right.

Number of pins on the perimeter of the figure: ____

Area of the figure from your formula: _______

Area of the figure by counting square units: ______

Do the results agree? __________



Second Investigation:

7. On your geoboard, form some convex polygons that contain at least one interior point. Enter your data in the table at the right. You might begin with one of the simplest examples, a square of area 4 square units. Enter at least three examples.

area	pins on the perimeter	pins in the interior
4		

8. It may be harder to see a pattern here, as we are trying to relate the area, A, to two variables, the number of pins on the perimeter, P, and the number of pins in the interior, I. To complicate things further, there could be a constant term in the formula. In other words, our formula could be of the form,

$$A = m \cdot P + n \cdot I + k$$

where m, n and k are constants that we need to determine.

9. We know from our first investigation that when the area is 1 square unit there are 4 perimeter points. Enter this data in this table.

10. Also, if you did not do so already, form a diamond –shape on your geoboard that encloses just 1 interior point. Enter this data in this table.

11. Form a system of three equations using the data from your table. For example, using the data from the first line, the equation would be

$$4 = m \cdot 8 + n \cdot 1 + k$$

Write two more using the data from your table. (Note that the data acquired in steps 10 and 11 might make some simple equations, although any data will work.)

Equation #2:________________________ Equation #3 ___________________________

12. Now solve the three equations simultaneously and determine the values of m, n and k. Then verify that your formula works by trying several examples.

Population Growth

Materials needed: About 350 – 450 skittles or similar objects that have two sides. Plastic two-colored disks work. Also a flat box with a lid that is large enough to contain the objects, and to allow them to be one row deep.

Directions:

A new insect has been discovered in New Hampshire! They are lively little creatures called Wiggies. They reproduce asexually (by themselves). Reproduction is triggered when the underside of the Wiggie is exposed to light. Let's assume that we have found a population of three Wiggies living outside the school in an old dead tree. We want to simulate the reproduction process and thereby the growth of the population of this insect with the following experiment.

1. Place three Skittles, representing the three Wiggies in our original population, into a flat container with a lid.

2. Shake the container and then open the top and count the number of "s" marks you can see. For each S, add another Skittle (to represent a new Wiggie that has been produced), to the box. Record the number of Wiggies in the container as being the number alive after the first reproductive cycle.(We will assume none expired.)

3. Pass the box to the next person. Again shake the box and then remove the top to count the number of "s" marks showing. Add another Skittle for each "s" showing and record the number as the total population at the end of cycle 2.

4. Repeat this process for two more shakes, recording your results each time.

5. Now make a guess as to what the population of Wiggies would be after 8 more shakes (for a total of 12 shakes) .

My guess ______________

6. Continue the process until you have completed 12 shakes or have run out of Skittles. Be sure to record your results each time in the table.

7. Was your guess higher or lower than the results you obtained from the experiment?______

Compare your results with some classmates.

reproduction cycle	total population of Wiggies
0	3
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

8. Transfer your results to your calculator and make a scatter plot of the data. Does it look like you can model the scatter plot with a polynomial function? What might be a good hypothesis? Quadratic Cubic? Higher degree? How could we test our hypothesis?

____

9. The data we collected is experimental. This makes it hard to examine with our usual methods of forming equations or finding common differences. However, we can learn a bit more about the nature of this type of growth by creating some theoretical data. Do the following:

Enter 3 on your calculator screen. Let's assume that in each cycle, half the Wiggies will reproduce. Thus after the first cycle, we would have 3 + 0.5*3 or 4.5 Wiggies. (For the moment, keep the fractional portion of Wiggles, although this is not very realistic.) Type + 0.5*ANS to obtain the theoretical number for the second cycle. (This should be 6.75). Now we can *iterate* to simulate the remainder of the experiment by pressing ENTER 10 more times. Record your answer ___________
How close was it to our data? _____________

10. By looking at what we did here, we can obtain a function that models the theoretical data. Note that we began with $y_0 = 3$ and then formed $y_1 = 3*1.5 = y_0*1.5$. We then multiplied by 1.5 again to obtain $y_2 = 1.5*y_1 = 1.5*1.5*3 = (1.5)^2*3$. Continue this pattern until you have found a general formula for y_n. $y_n =$ ____________________

11. Is the function we found in step 10 a polynomial function? Polynomials have the property that if you take successive differences in the dependent variable, assuming the independent variable increase by one each time, eventually the successive differences will become constant. For example, the values for the cubic equation $y = x^3 - 2x^2 + 1$ have been entered in the table at the right for $x \in [0,6]$. In the first-differences column, place the successive differences of the y-values. In the second-difference column place the successive differences of the values in the first-difference column. Finally, place the successive differences of the second column in the third – difference column. Observe that as successive differences are taken, eventually the column contains a constant. Why do you think this occurred in the third-difference column? You might test your answer by trying the same procedure with a quadratic.

x	y	1st diff	2nd diff	3rd diff
0	1			
1	0			
2	1			
3	10			
4	33			
5	76			

12. Thus, we might look to see if our hypothetical data could be modeled by a polynomial. In your scatterplot spreadsheet, enter the hypothetical data in column 3 by adapting the formula we obtained in step 10 and typing $L3 = 3*(1.5)^{L1}$.

13. Is our formula a polynomial formula? Apply the $\Delta list$ function in the calculator to successive columns. What do you observe?

14. Finally, now assume that the population, initially 3 Wiggies, doubles each cycle. Make a chart on your paper for the first 6 cycles and then take the common differences of each successive column as in step 13. A function of the form $y = ab^x$ is called an *exponential function*. Describe the property of exponential functions that we have just discovered.

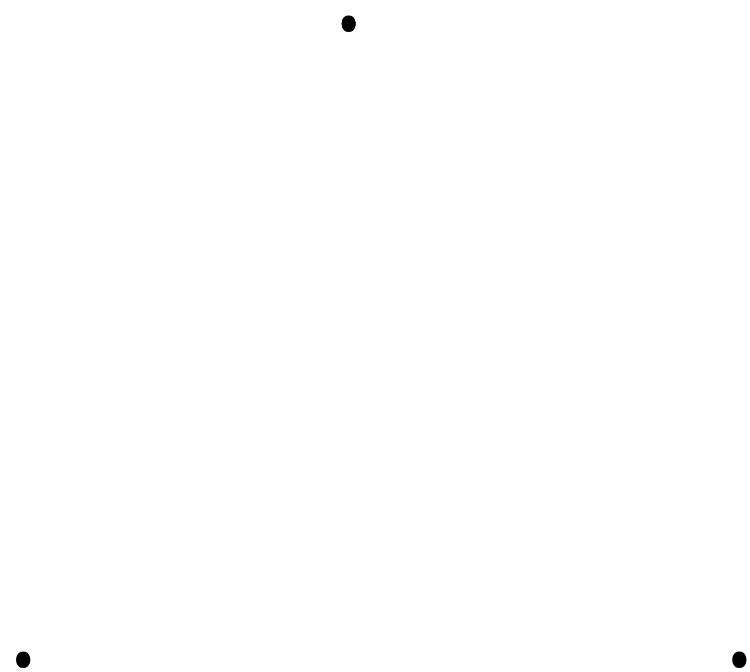
____

The Towers of Hanoi problem

Materials needed: 5 disks of varying sizes for each group. Small metal washers work well as well as plastic lids of various sizes.

> In the great temple of Benares beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high, and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate and the others getting smaller and smaller up to the top one. This is the tower of Brahma. Day and night unceasingly, the priest on duty transfers the disks from one diamond needle to another, according to the fixed and immutable laws of Brahma, which require that the priest must move only one disk at a time, and he must place these disks on needles so that there never is a smaller disk below a larger one. When all sixty-four discs shall have been transferred from the needle on which at the creation God placed them, to one of the other needles, tower, temple and Brahamans alike shall crumple into dust, and with a thunderclap the world will vanish.
>
> Quoted by George Gamow in his book, <u>One, Two Three . . . Infinity</u>

1. Take two disks and place them on one of the dots above (which represent the points of the diamond needles) so that the larger is beneath the smaller. Following the rules outlined in the problem, what is the least number of moves it would take to transfer the two to another dot (needle)?



2. Do the same thing with 3, 4 and 5 disks, recording the least number of moves it takes in each case in the table at the right.

# of disks	# of moves
1	
2	
3	
4	
5	
6	

3. Do you see any patterns emerging? How many moves would it take to move 6 disks? ______ 7 disks? _______

4. Plot these values in your calculator and graph. Give an algebraic reason why this is not an example of a linear function.

 ____

5. Determine whether this is an example of a quadratic function. Be sure to either find the equation of the function or give a reason why it is not.

6. This function is different from the polynomial functions we have been studying. We need to look at a different model. Try your own ideas. If you need a hint, look at the bottom of the page.

My equation: ______________________

7. If we were to assume that a priest could move a disk every 1 second, can you predict how long the world will last? Assuming that the priests began moving these disks about one-half million years ago, is it worthwhile doing tonight's homework? planning for college? looking forward to retirement?

Hint: (add 1 to each of the values in the second column, and look for a pattern for these numbers.)

Geoboard Slope Activity I

Materials needed: each pair or person needs one 11-pin geoboard or arrange four 7-pin geoboards so that they represent the four quadrants of the *x*-*y* coordinate plane.

Directions:

Put rubber bands vertically and horizontally through the center pin to represent the *x* and *y*-axis. The individual pegs of the geoboards represent points with integer coordinates. These points are known as *lattice points*, or *grid points*. In this activity, the student will take the end points of the rubber bands and place them on the correct pegs so that a line segment is represented by the rubber band. For the purpose of this activity, let's assume that the left most end point of the rubber band line segment is called *A*, and the other end point is called *B*.

1. **Some general questions.**

a) Create segments with **different** positive slopes, one in each quadrant, by placing end points of rubber bands in appropriate locations. Record the coordinates of the end points of the rubber bands *A* and *B* and the slope of *AB* in the table shown. Write a description of segments that have positive slope in the space below.

Quadrant	I	II	III	IV
Point *A*				
Point *B*				
Slope of *AB*				

____

____

b) Create segments with **different** negative slopes, one in each quadrant, by placing end points of rubber bands in appropriate locations. Record the coordinates of the end points of the rubber bands *A* and *B* and the slope of *AB* in the table shown. Write a description of segments that have negative slope in the space below.

Quadrant	I	II	III	IV
Point *A*				
Point *B*				
Slope of *AB*				

____

____

c) How would you describe lines that have zero slope? Write the coordinates for *A* and *B* for three different examples of line segments with zero slope.

____________________ ____________________ ____________________

d) How would you describe lines that have undefined slope? Write the coordinates for *A* and *B* for three different examples of line segments with undefined slope.

____________________ ____________________ ____________________

2. Remove all the rubber bands other than the *x* and *y* axis from the previous section of this activity. Put one end of a rubber band on the indicated point *A*, and the other end on a point *B* somewhere on the board so that the slope of the line *AB* is the number in the left column of the table below. (Assume the restriction that B must be to the right of A no longer applies.) Note that there may be more than one solution. Record as many as you are able to in the indicated space.

	Slope	**Coordinates of *A***	**Coordinates of *B***
(a)	$\frac{2}{1}$	(0,0)	
(b)	$\frac{1}{2}$	(–4,3)	
(c)	$-\frac{3}{1}$	(–1,1)	
(d)	$\frac{2}{3}$	(3,2)	
(e)	Undefined	(3,1)	
(f)	$-\frac{3}{4}$	(–1,1)	
(g)	$-\frac{3}{2}$	(-3,2)	
(h)	$\frac{5}{3}$	(4,1)	

Geoboard Activity for $y = mx + b$

Materials needed: each pair or person needs one 11-pin geoboard or arrange four 7-pin geoboards so that they represent the four quadrants of the *x*-*y* coordinate plane.

Directions:

Put rubber bands vertically and horizontally through the center pin to represent the *x* and *y*-axis. The individual pegs of the geoboards represent points with integer coordinates. These points are known as *lattice points*, or *grid points*. In this activity, the student will place the endpoints of the rubber band on grid points *A* and *B* so that the rubber band will be on the line whose equation is given in slope-intercept form in the left column of each table.

I. In this section, one of the endpoints of the rubber band is always the origin (0,0). Determine the coordinates of point *B* for the other endpoint of the rubber band, and record its coordinates in the table below. Note that there may be more than one solution. Record as many as you are able to in the indicated space.

	Equation of Line	**Coordinates of A**	**Coordinates of B**
(a)	$y = 4x$	(0,0)	
(b)	$y = -3x$	(0,0)	
(c)	$y = \frac{3}{2}x$	(0,0)	
(d)	$y = 0$	(0,0)	
(e)	$y = -\frac{5}{4}x$	(0,0)	
(f)	$x = 0$	(0,0)	
(g)	$y = \frac{4}{3}x$	(0,0)	
(h)	$y = -\frac{2}{5}x$	(0,0)	
(i)	$y = \frac{2}{3}x$	(0,0)	

II. In this section, the problems are more challenging than in part I. Not only do you have to determine both endpoints *A* and *B* for the rubber band to be on the given line, but none of these lines contain the origin. Determine the coordinates of both *A* and *B*, and record their coordinates in the table below.

	Equation of Line	**Coordinates of A**	**Coordinates of B**
(a)	$y = 2x - 3$		
(b)	$y = -1.5x + 4$		
(c)	$y = 2$		
(d)	$y = \frac{7}{4}x - 3$		
(e)	$x = 4$		
(f)	$y = -\frac{4}{5}x + 1$		
(g)	$y = \frac{4}{3}x - 5$		

III. This section is the most challenging of all because the *y*-intercept is not the origin as in part I, and it is not even an integer as in part II. Think carefully, you can do it.

	Equation of Line	**Coordinates of A**	**Coordinates of B**
(a)	$y = \frac{3}{2}x - \frac{5}{2}$		
(b)	$y = -0.6x + 1.8$		
(c)	$y = \frac{1}{4}x - \frac{3}{4}$		
(d)	$y = \frac{4}{3}x + \frac{5}{3}$		
(e)	$y = 0.375x - 0.5$		

IV. If your geoboard was infinite in its dimensions, would the line $y = 1.2x + 1.5$ contain any lattice points? State the coordinates of such a point, or explain why none exists.