Using A Hands-On Activities Based Approach in Algebra 2 and Beyond

Exeter Mathematics Institute

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Bug Jumping Investigation

Materials needed are graph paper, pennies, calculator

1. A bug jumps from lattice point to lattice point on a piece of graph paper, one jump per second, according to the following pattern: From (m, n), the bug jumps only to $(m+1, n)$ or $(m, n+1)$, each equally likely. Find all the places the bug could be, two seconds after it leaves the origin $(0, 0)$.

(0,0)
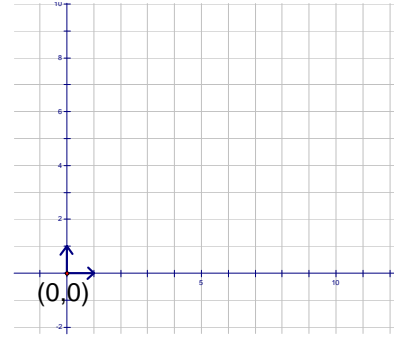
2. An interesting question is whether or not all the places the bug could land are equally likely? In order to help answer this question, we will simulate the twice-jumping bug by tossing two coins. Heads will equate with the bug jumping *right* and tails with the bug jumping *up*. Toss two coins 20 separate times, and record your results in the first row of the table below. A result of 2 heads means the bug landed at (2,0), 1 head and 1 tail equates with the location (1,1), and 2 tails means (0,2). Also put your results in the class table on the blackboard. Once all the groups have reported, add up the class totals and put them into the 2nd row of the table.

Bug Location	(0,2)	(1,1)	(2,0)
Frequency out of 20 trials from our group			
Frequency from total class trials			
Calculator simulation of 320 trials (< 1 min)			

3. Coin tossing is certainly a fun and legitimate way to simulate a random event that has two equally likely outcomes. A more efficient way to simulate many trials is to use a computer or calculator and their built in random number generating feature. Take your calculator to the instructor in order to have a bug jumping simulation program transferred into your calculator. The instructor will explain to the class how to use the program so that you can fill in the final row in the table above. If you have some extra time, try and run a simulation with as many as 1000 trials.

4. Now it is time to look at the data in the table and to analyze what is going on. Clearly, the three possible locations for the bug after two jumps from the origin are not equally likely. Explain below why (1,1) seems to be about twice as likely as either (2,0) or (0,2).

5. Since there is only one path to the points (0,2) and (2,0), but 2 paths to (1,1), the numbers 1,2,1 can be placed on top of those lattice points as shown at right to illustrate this important fact.

2

1

(0,0)
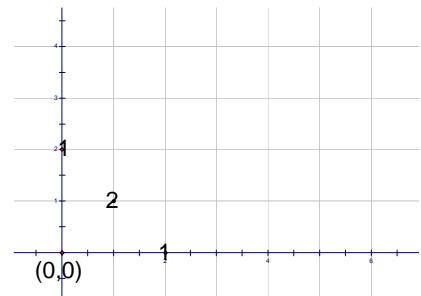
6. You are now going to repeat the same simulations with coins and the calculator for three jumps rather than two. The table below should be completed in the same manner as the one on the previous page.

Bug Location	(0,3)	(1,2)	(2,1)	(3,0)
Frequency out of 20 trials from our group				
Frequency from total class trials				
Calculator simulation of 640 trials (approx 2 mins)				

7. Once again, all the possible locations are not equally likely. Where the three locations from the 2-jump problem were in the ratio 1:2:1, what seems to be the ratio of the four possible locations in the 3-jump problem? Explain the reason for this ratio in the space below.

8. Similar to the previous example, we want to add the ratio numbers 1:3:3:1 that you found above on top of the appropriate lattice points on the coordinate grid as shown at right.
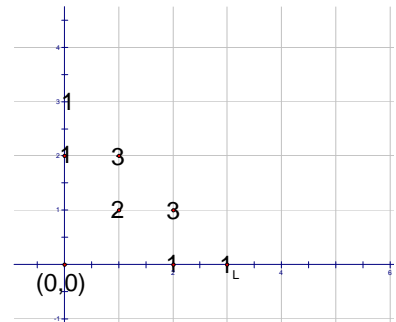
9. Let's do one more simulation, but this time let's go right to the calculator to shorten the process. By combining our calculator results, we should be able to simulate thousands of trials and obtain reasonably accurate information about a bug jumping 4 times. Each group should run 500 trials on their calculator for 4 jumps, and then record their results both in the table below and on the board.

Bug Location	(0,4)	(1,3)	(2,2)	(3,1)	(4,0)
Calculator simulation of 500 trials (approx 2 mins)					
Frequency from total class calculator trials					
Location ratio using 1 for both (0,4) and (4,0)	1				1

10. Explain below the reason for the ratio in the 4-jump problem, and add those ratio numbers on top of the appropriate lattice points on the coordinate grid. It should now look like the one at right. Hopefully you are starting to notice some patterns emerging. Write down any observations you have that might prove helpful if you were to try and predict what would happen in a 5-jump problem.
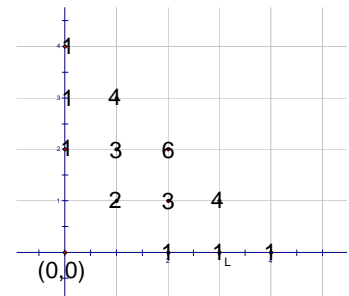
11. Multiply out $(r+u)^2, (r+u)^3$, and $(r+u)^4$, looking for connections with the bug-jumping problem. In algebra it is customary to collect terms like *rru*, *rur*, and *urr* into a single term $3r^2u$.

$(r+u)^2 =$

$(r+u)^3 =$

$(r+u)^4 =$

12. Use the patterns you have observed to write down the expansion for $(r+u)^5$ without doing any additional multiplication or algebra.

13. Explain below why it makes sense that the number of paths to the point (3,2) from the origin will equal the sum of the number of paths from (3,1) and (2,2).

14. Use the knowledge you have gained from this exercise to extend the triangular pattern below, both above and below the numbers that have come from the 2-jump, 3-jump, and 4-jump cases.

<div style="text-align:center">

      1   2   1

  1   3   3   1

1   4   6   4   1

</div>

Advanced – Without extending the table to the 20th row, try and determine the coefficient of the r^8u^{12} term in the expansion of $(r+u)^{20}$.

A Cone Investigation Using M&M's

Materials needed: Cardboard stock or thicker paper with an 8 or 10 cm circle drawn on it, protractors, scissors, scotch tape, ruler, 1 lb of M&M's for every 3 teams, and a graphing calculator.

The Problem: Determine the maximum volume right circular cone that can be formed from a sector of the given circle. Also determine the central angle of the sector that forms the maximum volume cone.

Step 1: After distributing the paper with the circles, every student or team of students is assigned an angle from the chart on the next page. The team's first job is to cut out a sector from the circle with the central angle that has been assigned to them.

Step 2: Form the sector of the circle into a cone and use the scotch tape to secure it. Now, fill the cone with M&M's and determine the "M&M volume" of the cone. Especially for the cones that have a large radius, try and be careful to fill the cone completely, but not to go way over. Record your number of M&M's in your table on the next page.

Step 3: Determine the height and radius of your cone using two different methods. First, use the ruler to measure these quantities in cm, and then record your estimated or measured height and radius in the appropriate cells in your table.

Step 4: Using the known radius and central angle of the sector that formed the cone, and the Pythagorean relationship between the slant height, height, and radius of the cone, calculate an exact height and radius for your cone. If this number differs substantially from your measured data, try and find your mistake. Record the calculated data in your table.

Step 5: Using your calculated height and radius, determine the actual volume of your cone in cubic centimeters. Record the volume data in your table. Now go to the board and copy all your data onto the master table for the class.

Step 6: When the master data table is complete for the entire class, copy all the data into your table.

Step 7: Using your graphing calculator, enter the column of angle data in L1, M&M volume data in L2, and the actual volume data in L3. Do a scatter plot of L1 vs L2 and L1 vs L3. Looking back on the original problem we are investigating, make some observations in the space below. A class discussion will follow.

Central angle of sector in degrees	M&M Volume	Height of Cone		Radius of Cone		Actual Volume
		Measured	Calculated	Measured	Calculated	
30						
50						
70						
90						
110						
130						
150						
170						
190						
210						
230						
250						
270						
290						
310						
330						
x						

Step 8: Notice that the last row has an angle labeled x. Go through the exact same process you used to determine the actual height, radius, and volume of your cone in order to determine an expression in terms of x for the volume of a cone with central angle x. The Pythagorean Theorem will of course play a central role. Use the space below for your work.

Step 9: By entering the expression you determined above into $y1$ in your graphing calculator, you will be able to use either the calculators table or graphing capability to decide if your volume function is correct and indeed fits the data collected by the class. Do this, and find and correct any errors that you detect.

Step 10: Using the Maximum feature of the calculator, determine an even more accurate answer to the question we are investigating than the one you came up with in step 7. How do they compare?

Bonus Question: When a sector is cut from a cone, another companion sector is also formed. What sector and its companion have the most combined volume?
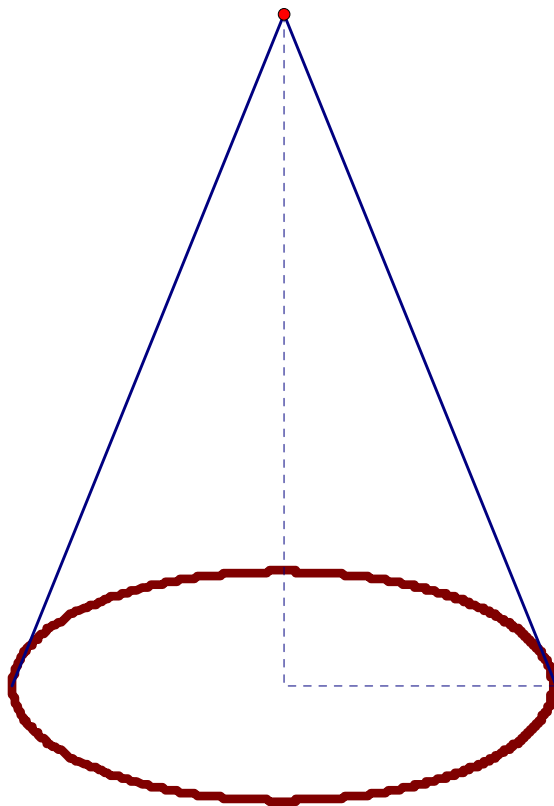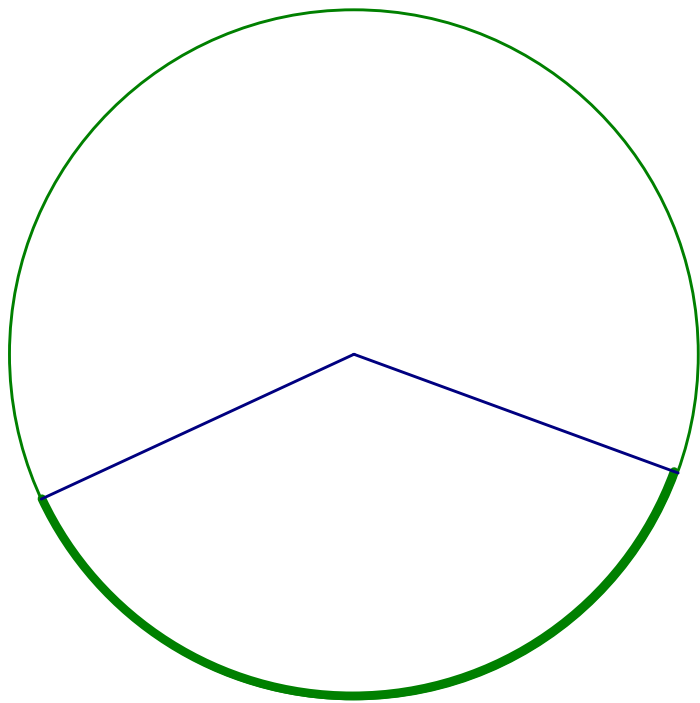
Car Intersection Investigation

Materials needed are graph paper, calculator, ruler, and a colored pencil or pen.

Two cars are traveling towards an intersection at a constant speed and direction. Car A is traveling east at 45 mph and reaches the intersection at 3:00 p.m. Car B is traveling north at 60 mph and passes the same intersection 25 minutes later at 3:25 p.m. Question 1: To the nearest minute, when were the cars exactly 40 miles apart? Question 2: When did the cars come closest to each other, and what was the time of day?

1. Set up a model of this problem on a piece of cm graph paper using the origin at the center of the paper as the intersection. Use a scale of 1 cm for every 10 miles for the coordinate grid so that the car's location between 2 p.m. and 4 p.m. will appear nicely on the graph. Use two different colors to represent the 2 cars, and place clearly labeled points as to their location at 3 p.m.

A at (0,0) 3 p.m.

B at (0,-25) at 3 p.m.
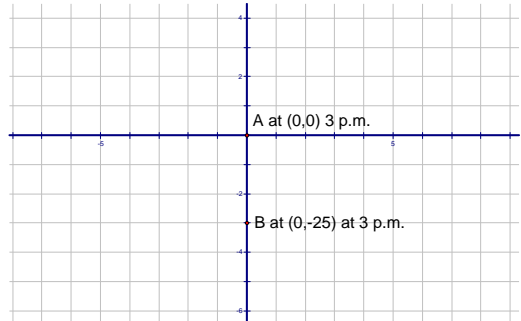
2. Answer the following questions in order to help you with task #3. How fast is Car A and Car B traveling in miles per minute? How far does each car travel in 12 minutes?

3. Fill in the first two columns of the chart below. This will track each car's location every 12 minutes between 2 p.m. and 3 p.m.

| Time of Day	Car A's coordinates	Car B's Coordinates	Measured distance between cars in cm (ruler)	Actual distance between cars in miles (distance formula)
2:00				
2:12				
2:24				
2:36				
2:48				
3:00	(0,0)	(0,–25)		
3:12				
3:24				
3:36				
3:48				
4:00				

4. Plot the locations carefully on the graphical model of this problem. Label the point's coordinates and time of day for each data point.

5. Using the cm scale on a ruler, carefully measure the distance in cm between the cars and record those measurements in the chart on the previous page.

6. Recall that the distance formula between two points (a,b) and (c,d) is $\sqrt{(a-c)^2+(b-d)^2}$. Use this formula and your calculator to calculate to 2-decimal place accuracy the distance in miles between the cars for each data point in the table. Record your results.

7. From just the data in the table, give your best estimate to the answers to the two questions that were stated in the original problem.

8. Let's try and find a more accurate answer to these questions by letting the variable t represent the time in minutes past 3:00 p.m. (Note that a negative t value is perfectly OK and simply represents time prior to 3:00 p.m.) Recall from task #2 that Car A travels at 0.75 miles per minute and Car B travels at 1 mile per minute. Explain in the space below why coordinates of Car A and Car B in terms of t are respectively Car A = $(0.75t,0)$ and Car B = $(0, t-25)$.

9. Write an expression in terms of t for the distance between Car A and Car B.

10. In order to enter this expression as a function into the calculator, it is necessary to change the letter t to an x. Enter this function into y_1 in the calculator, and then go to TBLSET by typing 2nd WINDOW. Type in -60 for TblStart, and 12 for ΔTbl. Then go to TABLE and you should see the same data that appears in the last column of the distance chart on the previous page.

11. Change the TblStart and ΔTbl parameters in order to make a better approximation to the two questions in this problem. You should be able to make a significant improvement on the answers from task #7.

12. Choose an appropriate window and have the calculator graph the distance function in y_1. By also graphing $y_2 = 40$ and finding the intersection on the calculator, you should be able obtain a very accurate answer to Question 1. By using the CALC menu (typing 2nd TRACE), option 3 will enable you to find the minimum of y_1 and thus answer Question 2 very accurately. Write both answers below.

Parabola Investigation

Materials needed are cm graph paper, calculator, compass, and ruler.

1. Using cm graph paper in a vertical position, put the *y*-axis in the center of the paper, and the *x*-axis a few lines up from the bottom of the graph paper. Put a point at coordinates (0,2) and label it *F*. Also plot the point *N* = (4,0). Draw line segment *NF*, and your sketch should look like the one at right.
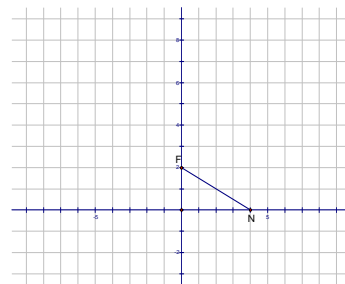
2. Using the compass and straight edge construction techniques learned earlier; construct the perpendicular bisector of segment *FN*. What is true about all the points that lie on the perpendicular bisector of line *FN*?

3. Since the graph paper already has vertical lines displayed, it is easy to use the straight edge to pencil over the perpendicular line to the *x*-axis that passes through the point *N* = (4,0). Label the intersection point of this line and the perpendicular bisector of *FN* as point *P*. In this instance, point *P* happens to be a lattice point, so its coordinates are easily discernable. Enter them into the table below #4 in the appropriate row. Note that point *P* is not only equidistant from points *F* and *N*, but it also has the important property that it is equidistant from point *F* and the *x*-axis.

4. It is now time to go to the instructor and to get assigned a new point *N* from the table below. Once this new point *N* has been assigned to you, you will repeat exactly what you did in tasks #2 and #3 above. That is, you will construct with compass and straight edge the perpendicular bisector of *FN*, and then you will mark the point of intersection *P* of this perpendicular bisector with the vertical line through *N*. Estimate the coordinates of this intersection point as accurately as you can, and enter it into the table below and also on the board.

Point *N*	(1,0)	(2,0)	(3,0)	(4,0)	(0.5,0)	(6,0)	(1.5,0)	(8,0)	(−1,0)	(−2.5,0)	(−4,0)	(−5.5,0)	(−8,0)
Estimate of Point *P*													

5. You will now use algebra to determine the exact coordinates for the points you just estimated. Let's do point (4,0) first, and then you will repeat the algebraic process for the point you worked on in task #4. To write the equation of the perpendicular bisector of segment *FN*, we need the midpoint and the slope of *FN*. Recall that the slope of a line perpendicular to *FN* will be the negative reciprocal of the slope of *FN*. The coordinates of the midpoint of *FN* are _____? The slope of segment *FN* is _____?

   The slope of the perpendicular line to *FN* is _____? Write the equation of the perpendicular bisector of *FN* in calculator ready point-slope form, $y = m(x - h) + k$ in the space below. Since the vertical line through *N* has the equation $x = 4$, it should be easy to find the exact intersection point *P* by the method of substitution. In this particular case, it is a lattice point, so your algebraic answer should agree with what you found by construction and what is already in the table above.

6. Now repeat step #5 above for the point you were assigned previously. Do all the algebra in the space below, and record your results in the table below and on the blackboard.

Point N	(1,0)	(2,0)	(3,0)	(4,0)	(0.5,0)	(6,0)	(1.5,0)	(8,0)	(−1,0)	(−2.5,0)	(−4,0)	(−5.5,0)	(−8,0)
Exact coordinates of Point P													

7. Once you have copied the coordinates from the board into the table above, enter the coordinates into the calculator and do a scatter plot of this data. Remember the important property that is true about each of these points, they are all exactly the same distance away from the *x*-axis as they are from the point F (0,2). In the space below, make some observations about what you see, and also indicate any additional points that are not in the table above that we easily could add to our data without doing any more work.

8. The *locus* of points that are equidistant from a fixed point and a fixed line are on a curve called a **parabola**. The fixed point is called the **focus**, and the fixed line is called the **directrix**. The curve going through the points we have plotted so far has an equation that we are now going to find. If we call any point on the curve $P = (x,y)$, then validate that the distance from P to F is $\sqrt{x^2 + (y-2)^2}$, and the distance to the *x*-axis is $|y|$. The equidistant condition then yields the equation $\sqrt{x^2 + (y-2)^2} = |y|$. In the space below, square both sides of this equation, collect terms, and solve for y. Enter this function into y_1 in the calculator, and then graph.

9. Use algebra to find the intersection of the perpendicular bisector line you found for the point N assigned to you in step #6 and the curve $y = \dfrac{1}{4}x^2 + 1$ that you should have found in step #8. How many times does the line intersect the parabola _____ ? What is the relationship of this line to the curve?

10. Go to the instructor and get assigned one of the parabolas in the table on the next page. Use the equidistant relationship to write an equation for your parabola, simplify the equation so that *y* is isolated, and then record your equation in the table below, and on the table on the board.

Focus	(0,1)	(0,2)	(0,3)	(0,4)	(0,0.5)	(0,−0.25)	(0,−5)	(0,−1.5)	(0,−2.5)
Directrix	$y=-1$	$y=-2$	$y=-3$	$y=-4$	$y=-0.5$	$y=0.25$	$y=5$	$y=1.5$	$y=2.5$
Equation									

11. Looking at the data in the table above a relationship between the coefficient a in the parabola equations $y = ax^2$ and the distance p between the vertex and the focus should be readily discernable. Write that relationship in the space below.

12. Use the relationship above to determine the focus and directrix for the following parabolas.

$$a)\ y = x^2 - 2x + 1 \qquad b)\ y = -\frac{1}{8}x^2 - \frac{1}{2}x - 4 \qquad c)\ y = \frac{1}{2}x^2 - 2x + 2$$

13. a) Find the equation of the tangent to the parabola 12/a at the point (2,1).

   b) Find the equation of the tangent to the parabola 12/b at the point (4,−8)

14. Determine the equation for the following parabolas.

   a) Focus (−2, 6), Directrix $y = 0$ 　　　　　　　　　　　　b) Focus (4, 2), Vertex (4, 1)

Meeting at the Grill

Materials needed are cm graph paper, calculator, and ruler.

Pat and Kim are each in the habit of taking a morning coffee break in the Grill. Each one of them arrives at a random time between 9 am and 10 am, and stays for exactly ten minutes. Find the probability that Pat and Kim see each other on any given day.

1. We are going to use a graph to model this problem. Using a piece of cm graph paper oriented horizontally, put the origin near the lower left hand corner of the paper, and use a ruler to draw the *x* and *y* axis. Each axis will represent the time between 9 am and 10 am that Pat and Kim arrive at the Grill. Let Pat's arrival time run along the *x*-axis, and Kim's arrival time runs along the *y*-axis. The origin represents 9 am for both, and let each ruled line equate with 5 minutes. Therefore, each coordinate point in the 60-minute by 60-minute square region in the first quadrant represents an arrival time for both Pat and Kim. The graph below shows the region. The point plotted at *A*(20,40) represents an arrival time of 9:20 for Pat and 9:40 for Kim, while the point plotted at *B*(15,10) represents an arrival time of 9:15 for Pat and 9:10 for Kim.
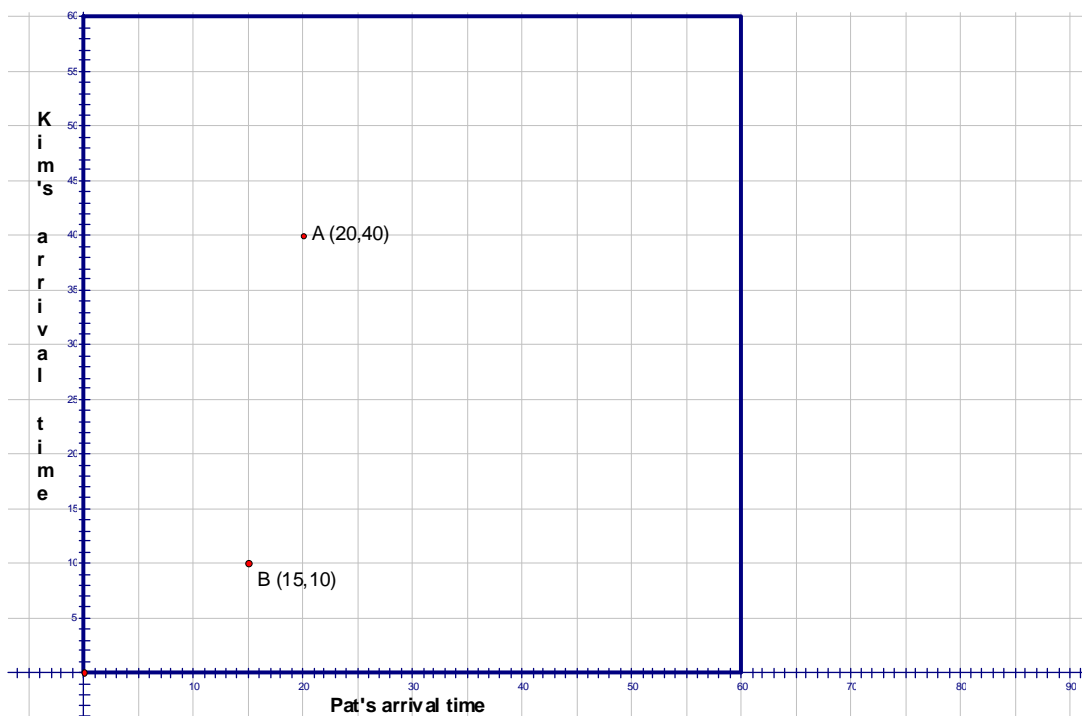
2. One of the two points above represents a situation in which Pat and Kim see each other, while the other one represents a situation in which they miss each other. Which one is which?

3. If you said they meet at (15,10), but miss at (20,40) you are exactly right. You and your partner are now going to use your calculators to simulate randomly Pat and Kim's arrivals. Using the **CATALOG**, enter **randint(0,60)** on the home screen. Notice when you hit enter, an integer between 0 and 60 (inclusive) appears on the screen. Hit enter again, and a new random integer between 0 and 60 appears. Notice you do not have to reenter the **randint(0,60)** command.

4. Now it is time for you and your partner to simulate one month's (30 days) worth of arrivals. Each of you will press the enter button on the calculator and generate an arrival time. One of you will be Pat and the other Kim. Record your individual results on a separate piece of paper by just writing down the coordinate pair. Next to each coordinate pair you write down, also write a *y* for when they meet or an *n* for when they do not meet. When finished, plot on your graph only those points that corresponded with a *y* for when they met. Then go to the big graph on the board and plot those points as well. Also write on the board the total number of points your group plotted out of your 30 trials.

5. To get a rough idea of the probability of Pat and Kim seeing each other, what summary information would you like to know from all the groups in the class?

    If you said the total number of days we simulated and the number of times our simulation indicated Pat and Kim saw each other, then you are exactly right. Gather that information from the board and do the appropriate division below to calculate a rough estimate of the probability they will see each other.

6. Looking at the graph on the board and on your paper, you should be able to locate the region where points must fall in order for Pat and Kim to see each other. Outline accurately this region that indicates they meet. For advanced students, try and write an inequality or set of inequalities that describes this region. Now, use areas of appropriate regions to find an exact probability of Pat and Kim seeing each other.
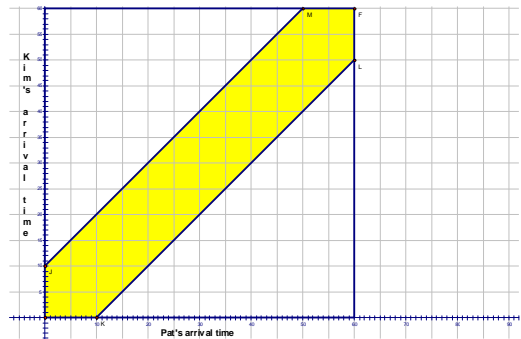
Extra Credit Opportunities

1) To the nearest minute, find how long Pat and Kim would each need to stay at the grill in order for the probability of them seeing each other to be greater than 50%.

2) Program your calculator to simulate Pat and Kim meeting at the grill any desired number of times, and also have the program calculate the simulated probability of them meeting.

A Lattice Point Investigation

Supplies needed: Graph paper or geoboards depending on your choice.

Question to investigate: How many lattice points are inside the region to the right of the y-axis and whose coordinates are constrained by $x + y < n$ and $x - y < n$, where n is any positive integer?

1. Let's proceed by letting $n = 1, 2, 3,\dots$ and drawing an accurate picture of the region on graph paper and counting the lattice points to see if a pattern emerges. If using geoboards, proceed the same way, but simply use rubber bands to represent the lines $x + y < n$ and $x - y < n$. Depending on the size of the geoboards, you will need to use more than one in this investigation as n increases.

2. Complete the chart at right for the values of n shown and start to look for a pattern that will help determine an explicit formula for the number of lattice points as a function of n. Remember, a lattice point needs to be **inside** the region, not on a boundary in order to count.

3. Write down in the space below any pattern you notice, and try and predict what the number of lattice points will be for an n value of seven using your pattern.

n	# of Lattice Points
1	
2	
3	
4	
5	
6	

4. Hopefully you noticed that the number of lattice points was always a perfect square of the number one less than the value of n. Thus, the answer for the number of lattice points for the case $n = 7$ would be 36, for $n = 8$ it is 49, and in general it is $(n-1)^2$. You may have also noticed something else about the pattern for the increase in lattice points. Rather than the number of lattice points increasing by a constant each time (which would indicate a linear pattern), the increase itself went up by a constant amount of 2 each time. This is an important indicator that the nth term is a *quadratic* function of n. That means of the form $an^2 + bn + c$, where a,b, and c are constants. What are the values of a,b, and c for the example we just did?

5. Let's repeat the steps #1 - #4 from the previous page for a slightly different region. This time the region is defined as being to the right of the *y*-axis and constrained by $2x + y < 2n$ and $2x - y < 2n$. Once again, draw the region accurately on graph paper, or model it with a geoboard and rubber bands. Complete the chart at right, and compare with others in the class before moving to the next steps.

Note: Once again, you should notice that the increase in the increase in lattice points from one row to the next is a constant. Another way of saying this is that the 1st differences in this data set are linear, and the 2nd differences are a constant. This once again indicates that the *n*th term is a *quadratic* function of *n*. That means of the form $an^2 + bn + c$, where *a,b*, and *c* are constants. Unlike the last example in which the *n*th term had been easily detected as $(n-1)^2$, this time it will take some algebra to determine the *a,b*, and *c* constants.

n	# of Lattice Points
1	
2	
3	
4	
5	

6. In order to determine *a,b*, and *c*, set up a system of three equations and the three unknowns using three of the data points from the chart at right. This will enable you to algebraically solve the system for *a,b*, and *c*, and thus find a formula for the *n*th term.

7. Check your formula with the other known data points, and also use the *n*th term to determine the number of lattice points when $n = 6$ and $n = 7$. It should be 55 and 78 respectively.

8. Let's do one more before attempting a generalization. Repeat steps #1 - #4 from the previous page for a third region. This time the region is defined as being to the right of the *y*-axis and constrained by $3x + y < 3n$ and $3x - y < 3n$.

9. Fill in the chart at right, and see if you can see the generalization for the last row?

Region defined by inequalities below and to the right of the *y*-axis	Number of lattice points inside the region as a function of n in standard form $an^2 + bn + c$	Number of lattice points inside the region as a function of *n* in factored form $(x-1)(mx+b)$
$x + y < n$ and $x - y < n$		
$2x + y < 2n$ and $2x - y < 2n$		
$3x + y < 3n$ and $3x - y < 3n$		
$kx + y < kn$ and $kx - y < kn$		

The Cheetah Problem

Materials needed: graph paper, ruler, calculator, and compass

A cheetah can run at 105 feet per second, but only for 7 seconds, at which time the animal must stop and rest. A fully rested cheetah at (0,0) notices a nearby antelope, which is located at (−39, 228), moving linearly in the northeast direction at 50 feet per second along the line with slope ¾. Assume the cheetah does not change directions once it starts to run, and that it accelerates instantaneously to its top speed.

Question: How many seconds can the cheetah afford to wait before starting to run? If the cheetah waits the maximum amount of time, where and when will the cheetah intercept the antelope?

1.   Start with a piece of graph paper in horizontal position and locate the origin a few inches from the lower left corner. Using a scale of 20 units for each graph paper line (50 units per line if cm graph paper), plot the antelope's and cheetah's initial positions at time zero. Since we know the antelope travels on a line with slope 3/4, plot another point on the antelope's path and draw a dashed ray representing the path of the antelope. Write the equation below of the line going through the point (−39,228) and having a slope of 3/4. Your picture should look like the one at right.

2.   After completing this step, each group should obtain one of the times in the chart below. They represent the amount of time that has passed from $t = 0$ as the cheetah waits to start to run. Once they have obtained a time, their job will be to determine the exact coordinates of the antelope for that time, the distance from that point back to the original location of the cheetah (0,0), and how long it would take the cheetah to travel to that point. Plot the coordinates for the antelope, and label it clearly on your graph. Also plot your information on the board. Let's do $t = 1$ together. After 1 second, the antelope will have traveled a distance of 50 feet (since its speed is 50 units per second) along a slope of ¾. This means for every 4 units to the right, the antelope moves 3 units up. This would be a total distance of 5 units by the Pythagorean Theorem (3-4-5 triangle). Therefore, to travel a distance of 50 units on a line with slope ¾, the antelope travels 40 units right and 30 units up every second. So its coordinates at $t = 1$ are (1, 258). Using the distance formula, the distance from the antelope at (1, 258) to the origin (0,0) is $\sqrt{(1^2 + 258^2)} = \sqrt{66565} \approx 258$ feet. Since the cheetah travels 105 feet per second, then it would take the cheetah only 2.46 seconds to reach that point.

Time in seconds after $t=0$	Coordinates of antelope	Distance from antelope to origin (0,0) (ft)	Time for cheetah to reach the antelope
$t=1$	(1, 258)	258	2.46
$t=2$			
$t=3$			
$t=4$			
$t=5$			
$t=6$			
$t=7$			
$t=8$			
$t=9$			
$t=10$			
$t=11$			
$t=12$			
$t=13$			
$t=x$			

4.   From studying the chart carefully, the answer to the question should become clear. What algebra equation could have been solved to answer the question had the answer not turned out to be such a nice number.

5.   Change the speed of the cheetah to 120 feet per second, keep everything else the same, and answer the question to 2-decimal place accuracy by using the calculator and algebra.

Bonus Question.   If the original 105 fps cheetah started to run at top speed immediately at $t = 0$, where and when would the cheetah intercept the antelope?

Triangle Area Optimization Investigation

Materials needed are cardboard stock cm graph paper, scissors, calculator, and ruler.

1. A line is drawn with negative slope that contains the point (2,5). This line intersects the *x*-axis at point *A*, and the *y*-axis at point *B*. Let *C* = (0,0), and the right triangle *ABC* is the object of this investigation. Specifically, what slope of the line passing through (2,5) will produce a triangle *ABC* with the least amount of area? Start with a piece of cm graph paper in the horizontal position, locate the origin in the lower left hand corner of the paper, and plot the point (2,5). Draw a line with slope −1 through the point (2,5), and your paper should look like the picture at right.

2. The equation for a line going through the point (2,5) with slope *m* is $y = m(x-2)+5$. In the space below, write the equation of the line with slope −1, and determine the *x* and *y* intercepts of this line. Label the intercepts *A* and *B* respectively, and also label the origin as point *C*. Determine the area of right triangle *ABC* and write that area clearly in the middle of the triangle. Now, use a scissors to cut out the triangle.

3. When you have finished step #2 above, it is time to go to the instructor and be assigned one of the slopes in the table below. Once you are assigned a new slope value, you will repeat step #2 using this new slope. Thus, you will draw a new line going through (2,5) with this new slope, write its equation, determine the *x* and *y* intercepts, and determine the area of triangle *ABC*. When finished, write the information in the table below and in the table on the board. Also, cut out the triangle and write the area in the center of the triangle.

Slope m	Equation of line	x-intercept A	y-intercept B	Area of triangle
−1	$y = -1(x-2)+5$	(7,0)	(0,7)	24.5
−0.25				
−0.5				
−1.5				
−2				
−3				
−4				
−5				
−6				
−7				

4. Copy the information from the entire class into your chart on the previous page, and also enter the data into your calculator. Do a scatter plot of the data in your calculator.

5. In the table below that looks exactly like the one on the previous page, the variable m is now in the first column for slope. Determine the entries in the other columns that will all be in terms of m.

Slope m	Equation of line	x-intercept A	y-intercept B	Area of triangle
m				

6. The expression for area in the last column gives us the area as a function of m. Enter this into the calculator in y_1. Note: You will need to use the variable x in place of m when entering the area function into the calculator. Once you have the function entered, look at a table of values for this function, and also graph the function. The window currently in the calculator from your scatter plot in #4 should be perfect to see the graph. Find the minimum value of the function, and also the x-value (which is now slope), that produces that minimum. What observations about the table and the graph can you make below? Is the function describing area a familiar function?

7. When you have reached this point, go to the instructor and obtain a new point P from the table on the next page to replace (2,5). Once you have been given a new point P, you will repeat steps #5 and #6 above to determine the minimum area triangle for a line going through that point and the slope of the line that causes that minimum. Enter your new information in your table and on the board.

Point P	Equation of line through P with slope m.	x-intercept A	y-intercept B	Minimum area of triangle	Slope of line that produces minimum
(2,5)	$y = m(x-2)+5$	$\dfrac{2m-5}{m}$	$-2m+5$	20	−2.5
(2,4)					
(2,3)					
(2,6)					
(2,1)					
(3,5)					
(3,6)					
(3,7)					
(4,2)					
(4,3)					
(a,b)					

8. Looking at the results in the table above, write observations in the space below.

The Box Investigation

1. Take the sheet of paper you have been given and cut off the white border so that just the graph paper remains. It should be 18 squares wide and 24 squares long.

2. Cut a square from each corner of the paper that is _____ squares by _____ squares. Your paper should then look like the picture at the right. The dotted lines outline the squares that were removed.
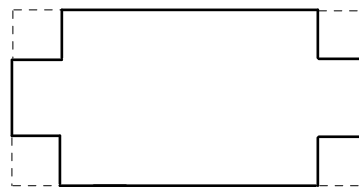
3. Crease your paper as accurately as possible along the four lines joining the cuts. There lines are dotted in the figure at the right.

4. Fold your paper along the creases you just made and tape the sides to form a box without a top.

5. Compute the number of Starbursts candies your box would hold and record it in the appropriate space in the table at the right.

6. Collect the other data for the table from your classmates and then enter the data into your calculator. Make a scatter plot from your data. Make a guess as to what type of equation might model these points.

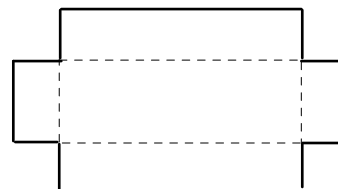| size of cut	volume of box in Starbursts
0 cm.	
1 cm	
2 cm.	
3 cm.	
4 cm.	
5 cm.	
6 cm.	
7 cm.	
8 cm	
9 cm	

7. Review how you found the number of Starburst that would fit in your box. Now let the size of the cut be x-cm. Imagine that you have cut out the squares and folded the box up. Represent in terms of x:

   The number of Starbursts that would fit along the length of the box _____

   The number of Starbursts that would fit along the width of the box _____

   The number of Starbursts that would fit along the height of the box _____

   The total number of Starbursts in terms in terms of x that would fit in the box _____

8. Let y_1 equal the expression that represents the total number of Starbursts, terms of x, and graph your equation. How well does model your scatter plot? What size cut would produce a box that would hold the most Starbursts?

9. What would be the size of the cut that would allow us to construct a box with maximum volume if you could use fractional parts of Starbursts? _____ What would be the volume of the box? _____

An Inscribed Area of a Circle Investigation

Materials Needed: cm graph paper, scissors, scotch tape, compass, protractor, straight edge.

The Problem: Find the maximum area of a right triangle that is inscribed in a semi-circle of radius 8 cm in which the right angle is on the diameter of the semi-circle.

The Investigation: **Part I**

1. Using the cm graph paper and the compass, draw a semicircle with a radius of 8 cm.

2. By assigning the point *A* which is the center of the circle the coordinates (0,0), point *B* will have the coordinates (*x*, 0). Go the instructor and get assigned one of the *x*-values in the table below.
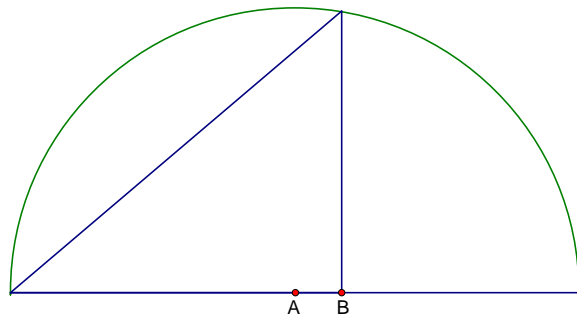
x	Base of triangle cm	Measured height of triangle cm	Calculated height of triangle cm	Area of triangle cm^2
−2				
0	8	8	8	32
1				
2				
3				
4				
5				
6				
7				

3. In order to complete the row of the table you were assigned, you will need to determine the equation of the circle in order to determine the calculated height of the triangle. Once you have completed the row of data you were assigned, enter it on the board. Also, cut out the triangle you were assigned, write its area on the front, and scotch tape it on the board in the appropriate location.

4. Once the table is complete for the entire class, copy the data into your own table; enter all the data into the calculator, and then do a scatter plot.

5. Add a row to your table, and this time let the first column entry be the variable x rather than a number. Write expressions in terms of x for the calculated height and the area of the triangle. Enter the expression for the area of the triangle into y_1 in the calculator, and graph this area function.

x	Base of triangle cm	Calculated height of triangle cm	Area of triangle cm^2
x			

6. Find the maximum of this area function, and also determine the dimensions and angles of this maximum area triangle.

7. Comment below on what you have discovered and why it makes sense.

8. An alternate approach would be to let the variable be one of the non right angles in the triangle and proceed that way. Try and write an area function of the angle instead of the x-value, and convince yourself you get the same result.

A String Investigation

Part I

1. Take a piece of string and place two knots on it 10 cm apart. Push a pin through each of the knots.

2. Take your graph paper and draw *x*- and *y*-axes so that the origin is in the middle of the paper. Mark the point $F_1 = (-3,0)$ and $F_2 = (3,0)$ on the paper.

3. Place the pins at F_1 and F_2. With one person anchoring the pins to the board, the other should use the pencil to pull the string taut. Keeping the string taut, drag the pencil on a tour that goes around the pins. It may be easier to do the top half first, then re-adjust the string and do the bottom half. The two halves should meet. Do this on two separate sheets so that each person has their own copy of the graph. It should be a closed curve, oval in shape. This is called an ellipse.
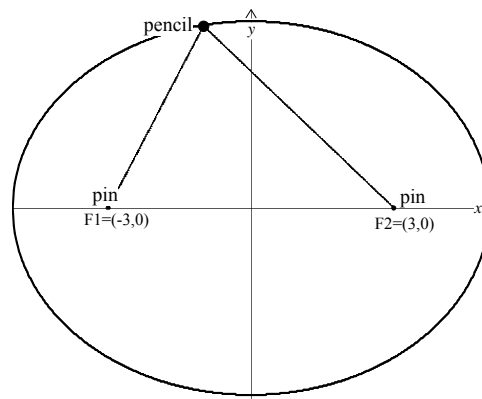
4. Before removing the pins, move the pencil point so that it is on the positive *y*-axis, keeping the string taut. Note the figure made by the string and the positive *x*- and *y*-axes. Knowing the placement of the pin and the length of the string, it should now be possible to compute the *y*-intercepts. Do so.

    Positive *y*-intercept: _____   Negative *y*-intercept: _____

5. Move the pencil so that it now rests on the *x*-axis, again with the string taut. Again, knowing the total length of the string, and the positions of the two points F_1 and F_2, compute the *x*-intercepts of the curve.

    Positive *x*-intercept: _____   Negative *x*-intercept: _____

6. Finally, move the pencil so that it now rests on an arbitrary point $P_{(x,y)}$ on the curve. Estimate the coordinates of the point *P*. Recall that PF_1 and PF_2 are the distances from *P* to F_1 and F_2 respectively, If the string is 10 cm. long, verify that $PF_1 + PF_2 = 10$.

    Estimated coordinates of P (to one decimal place) _____

    Distance $PF_1 =$ _____   Distance $PF_2 =$ _____   Total distance _____

28

Part II

7. Write the equation $16x^2 + 25y^2 = 400$ in a form that allows you to display it on a graphing calculator. (Note that you will actually need *two* equations.) Show your work below

First equation _____   Second equation _____

Your graph should confirm the *x*- and *y*-intercepts found in steps 4 and 5.

8. Refer to the equation in step 7. Divide each term by 400 so that the equation is in the form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. For your equation, what is the value of *a*? _____ of *b*? _____ What is the significance of these values to your graph?

9. Your work should provide you with an easy way to graph an ellipse. Consider the ellipse given by the equation $4x^2 + 9y^2 = 36$. By first converting it to the form in step 8, find the *x*- and *y*-intercepts and sketch the graph on the axis provided.
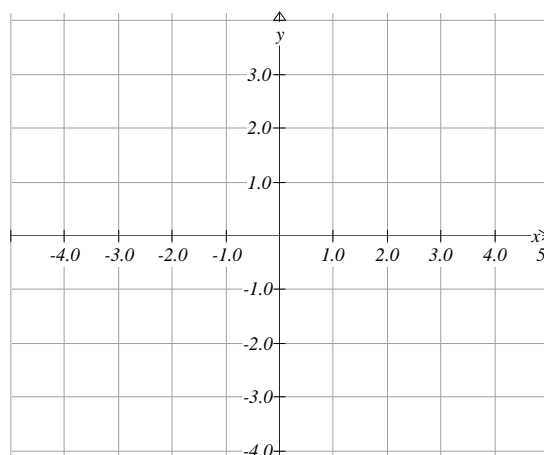
10. Solve the equation for *y* and graph on your calculator to confirm your work. Do you have the same intercepts?

11. Recall how you determined the *x*-intercepts in step 4. How long a string would be needed to draw this ellipse?

12. Using the technique from step 4 used to find the y-intercepts, find the location of the points where the pins must be placed to draw the ellipse. These points are called the foci (singular is focus) of the graph.

Part III

13. Let $F_1 = (-3,0)$ and $F_2 = (3,0)$. Also, let $P = (x,y)$ be any point on the ellipse drawn by using a 10 unit string. This is the same picture we started with in steps 1-3. Using the distance formula, write an expression for the distance

from P to $F1$: _____   from P to $F2$: _____

Now write the equation $PF_1 + PF_2 = 10$.

Simplify this result by removing the radicals from the equation. To do this, start by arranging the terms so that there is one radical on each side of the equal sign.

Now square both sides as in $(PF_1)^2 = (10 - PF_2)^2$. Remember to make use of the identity $(m-n)^2 = m^2 - 2mn + n^2$.

Simplify as much as possible and isolate the remaining radical on one side of the equation.

Square both sides again. If all goes well, you will eventually discover the equation $PF_1 + PF_2 = 10$ is equivalent to $16x^2 + 25y^2 = 400$.

Dice Simulation

Gather a number of six-sided dice (at least 50 or more) and a box which is large enough to contain these dice so they can be shaken. Follow the following directions. .

1. Select a number from 1 to 6.

2. Put all the dice into a box with a lid. Shake the box and then roll the dice out on a flat surface. A table top, floor or large tray can be used for this purpose.

3. Remove all the dice that show the number selected in step 1. Record the number of dice remaining on the chart below. (Increase the number of rows, if necessary.)

4. Put the remaining dice in the box and repeat steps 2 and 3 until no dice remain. HOWEVER, do not record the last roll that produces zero.

Roll number	Number of dice remaining
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

5. Enter the values from column 1 into LIST 1 and column 2 into LIST 2 of your calculator.

6. Generate a Scatter Plot of the data. Which pattern, linear, quadratic or exponential, does the data seem to follow?
     _____

7. Determine the regression equation you think fits best and enter it in Y1 (Round your decimals to two places). Write your equation here also.
                              _____

Answer the following questions concerning you data and graph.

8. What do the x and y represent in your equation? x _____

                                                                    y _____

9. In your equation, what does the constant that is raised to a power represent? _____

     _____

10. In your equation what does the initial constant represent? _____

     _____

11. Theoretically, what fractional part of the number of dice should have been removed each roll?

12. Compare the two equations and note any observations _____

---

13. Why did we exclude the roll that produces zero dice remaining? _____

     _____

14. How would your results be affected if a different die had been selected initially?

     _____

15. What would have changed if experiment had been done with two-sided coins? 12-sided dice?

     _____

     _____

A Lightning Strike Investigation

Materials needed: cm Graph Paper, Ruler Compass

Problem – Pat and Kim are talking on the telephone during a thunderstorm. After one of the lightning flashes, Pat hears the rumble of thunder *twice* – the first sound coming through the open window, and the second sound coming *through the telephone* ten seconds later. Given that Pat lives two miles east of the center of town, Kim lives two miles west of the center of town, both on the same east-west road, and that sound takes five seconds to travel a mile, draw a map that identifies some of the places where the lightning could have struck. For example, could the lightning have struck the road on which Pat and Kim live? For the purposes of this investigation, assume light and electricity take no time to reach their destination.

1. After drawing the coordinate axis in the middle of the graph paper, plot the points (–2,0) and (2,0), and label them *K* for Kim and *P* for Pat respectively. Draw a circle centered at *P* with a radius of 3 cm. Now draw a circle centered at *K* with a radius of 5 cm. Explain below why the points of intersection of these two circles are potential lightning strike locations.

2. Label the points of intersection of the two circles *A* and *B*. Although you may be able to ascertain the coordinates of *A* and *B* without algebra, it is important to review the algebraic steps in order to be able to repeat this process for less obvious examples later in this activity. The equation of the circle centered at Pat is $(x-2)^2 + y^2 = 9$ and the equation of the circle centered at Kim is $(x+2)^2 + y^2 = 25$. In the space below, do the algebra required to solve these equations simultaneously, and thus find the coordinates of points *A* and *B*.

3. Explain why the circle centered at Kim was constructed to be two greater in radius than the circle centered at Pat. What does this radius information tells us about this lightning strike problem?

4.   Go to the instructor to be assigned one of the rows in the table below. You will be assigned a radii for the circles centered at Pat and Kim, and your task will be to complete your row of the table by finding the equations of the circles and any points of intersection rounded to the nearest 2 decimal places. It is important that you draw the circles accurately on your graph as well in order to visually check your results. Once you have completed your row, add your data to the class chart on the board, and plot your two points carefully

Radius of Circle centered at Pat	Radius of Circle centered at Kim	Equation of Circle centered at Pat	Equation of Circle centered at Kim	Points of Intersection of the two circles
0.5	2.5			
1	3			
2	4			
3	5	$(x-2)^2 + y^2 = 9$	$(x+2)^2 + y^2 = 25$	$(2,3)$, $(2,-3)$
4	6			
5	7			
6	8			
7	9			
8	10			
9	11			

6.   Copy the data from the board into your own table, plot all the data points, and also enter the data into the calculator and do a scatter plot of the data.

7. If $L = (x, y)$ represents any point that is a possible lightning strike location, then explain which of the following equations makes sense. $LK - LP = 2 \; or \; LP - LK = 2$.

8. Using the distance formula, the equation $LK - LP = 2$ translates to:
$$\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = 2.$$
By moving one of the square roots to the right side and squaring both sides, you can start the process of isolating y. It will also take another squaring of both sides before you are done. Solve the above for y and graph accurately by entering the expressions for y into your calculator.

9. The curve described by this lightning strike problem is called a *hyperbola*. A hyperbola consists of all the points whose distances to two fixed points (focal points) differ by a constant. In this example, Pat and Kim were the focal points, and the fixed distance was two. An important characteristic of hyperbolas is their linear asymptotic behavior. What lines are acting as asymptotes in the problem above?

10. Try to find the equation of the hyperbola that models the lightning strike problem if we moved Pat and Kim to $(0, 5)$ and $(0, -5)$ respectively, and the first sound coming through the open window, and the second sound coming through the telephone were thirty seconds apart. Good luck.

A Pyramid Investigation

Materials needed: Any type of cubic candy that stacks easily, or dice or blocks will obviously work .

1. Start by building "cube" pyramids by stacking centered square layers. The dimension of each layer is one less than the dimension of the layer immediately below it. See picture at right. Let $S(n)$ stand for the total number of cubes required to build a pyramid whose bottom layer is n-by-n.
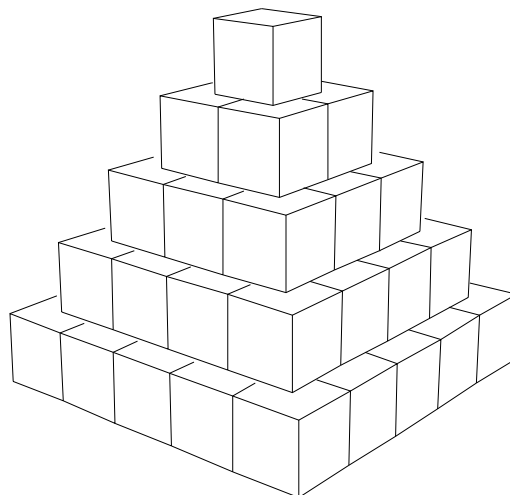
2. As you build pyramids of increasing size, fill out the chart below to help in our investigation to determine a formula for the total number cubes required in a pyramid whose bottom layer is n-by-n. This formula is what will go into the last row entry. As the dimension grows larger, it will not be practical to build the pyramid, but you should be able to decipher the number through a recursive process.

Dimensions of the bottom layer in the pyramid n x n	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6	7 x 7	8 x 8	9 x 9	10 x 10
Total Number of Cubes in the Pyramid $S(n)$										

3. After you have completed the chart, enter the data into the calculator and do a scatter plot.

4. As you look for a pattern of the numbers in the 2nd column of your chart, use the scatter plot and other techniques to try and determine if they fall into a familiar pattern you have experienced in other problems. These familiar patterns would include linear, quadratic, or exponential functions. Use the space below to summarize your conclusions as to which if any of those familiar functions might fit this data, or can easily be ruled out.

5. Hopefully you have ruled out linear, quadratic, and exponential functions from your work on the previous page. When analyzing whether or not the data was linear or quadratic, you should have looked at 1st and 2nd differences in the 2nd column to see if a constant emerged. Although neither 1st nor 2nd differences result in a constant, you should take note that 3rd differences **do** result in a constant. What conjecture can you make about the type of function that fits this data?

6. If you answered *a cubic polynomial* to the previous question, you are a star. This means that $S(n) = an^3 + bn^2 + cn + d$. Use four data points from your table to form a system of four equations and four unknowns in order to solve for *a, b, c,* and *d.*

7. Once you have determined the cubic function for $S(n)$, enter into the calculator and use the table feature to make sure it generates the same table of values as our pyramid data. Also, graph the function to see that it passes through the same points from the scatter plot.

8. Use the formula to express the volume of the pyramid as a percentage of the volume of an *n*-by-*n* cube. What is this percentage when *n* is a very large number? Explain below why this makes sense.

Another Pyramid Question

Materials needed: A ruler or meter stick, 1.5" Styrofoam balls (thirty balls needed for every group of participants), and toothpicks.

Question to Investigate: If the number of rows in the square pyramid of spheres pictured at right is *n*, what function of *n* describes the height of the pyramid. Assume the diameter is a constant *d*.
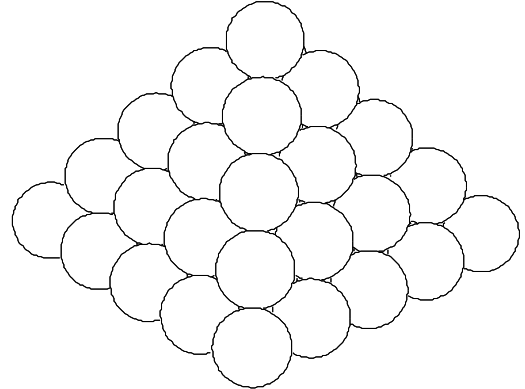
1. Measure the diameter of the sphere your group is working with.

2. Start building pyramids of increasing size, and measure the height of each pyramid as accurately as possible with a ruler or meter stick. Record the results in the 2nd row of the table below. You will only be able to build full pyramids up to the dimension 4 x 4. After that, do your best to estimate a couple more and record.

| Dimensions of the bottom layer in the pyramid *n* x *n*	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6
Measured Height of the Pyramid						
Calculated Height of Pyramid						

3. Enter your data into the graphing calculator and do a scatter plot using *n* as the *x*-variable. Looking at the data, make a conjecture about the kind of function that will model the height of the pyramid as a function of *n*.

4. If you conjectured that the height of the pyramid is a linear function with respect to the number of rows that was great. Use the capability of the calculator to determine a regression line of best fit for your data and record the equation of your line in the space below. Also, enter the equation into y_1 in the calculator and graph the line along with the scatter plot.

5. Now go to the instructor and get assigned a pyramid of a certain size. Your job will be to do the geometry to determine the actual height for your assigned pyramid. Once you have done the calculation, record your answer in your table and on the master table for the class that is on the blackboard.

6. Based on the work you have done above with your specific example; determine the exact function that gives the height of a square pyramid of spheres as a function of the number of rows n. Use d in the function to represent the constant diameter of each sphere

Another Pattern Investigation

In this exercise, try to discover the pattern to the following experiment. Once you have a conjecture, try to describe it mathematically either *recursively* and *explicitly*.

Recall that recursively means in terms of the previous term, while explicitly means finding a formula in terms of the number of that term. For example, in the sequence $u_n = 1, 3, 5, 7, 9, \ldots$ the sequence is described explicitly by $u_n = 2n - 1$. Thus if you want the 34th term of the sequence, you simply substitute 34 for n in the explicit formula. If the same sequence were described recursively, then the first term is given as $u_1 = 1$, and then the recursive description $u_n = u_{n-1} + 2$ says that any term in the sequence is the previous term plus 2. To find the 34th term, you need the 33rd term.

1. If you place one point on the circumference of a circle, you have not divided the interior of the circle. It is still one whole. However, if you place two points on the circumference anywhere and connect them, you divide the circle into 2 regions. Three points connected in all possible ways divides the circle into 4 regions. (Confirm this yourself.) Continue adding points and complete the table at the right below. Remember you want the <u>maximum</u> number of regions, so there should **never be more than two chords of the circle intersecting at any one location**. Use the circle below, and on the worksheet to come up with a conjecture.
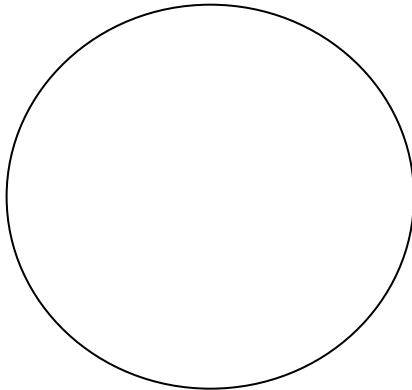
Number of points	Number of regions
1	
2	
3	
4	
5	
6	
7	
8	
9	

2. A pattern should appear by the fifth row. Use the space below to write down your observations.

3. Now try to write formulas that describe the pattern either explicitly or recursively.

4. Finally test your pattern by finding the number of regions formed by connecting six points. Does it work?

5. You should have discovered that the number of regions for six points produces the surprising result of 31 regions instead of what most people will predict is 32. This means that more investigating is necessary, and that the pattern is NOT going to be a simple exponential function. Construct and use a **large** circle to investigate further. The number of regions can become difficult to count as the number of points increases, so look for any patterns or helpful ways to make the job manageable. A sample of what the circle looks like with 9 points is given below. Good luck because this is a difficult problem.
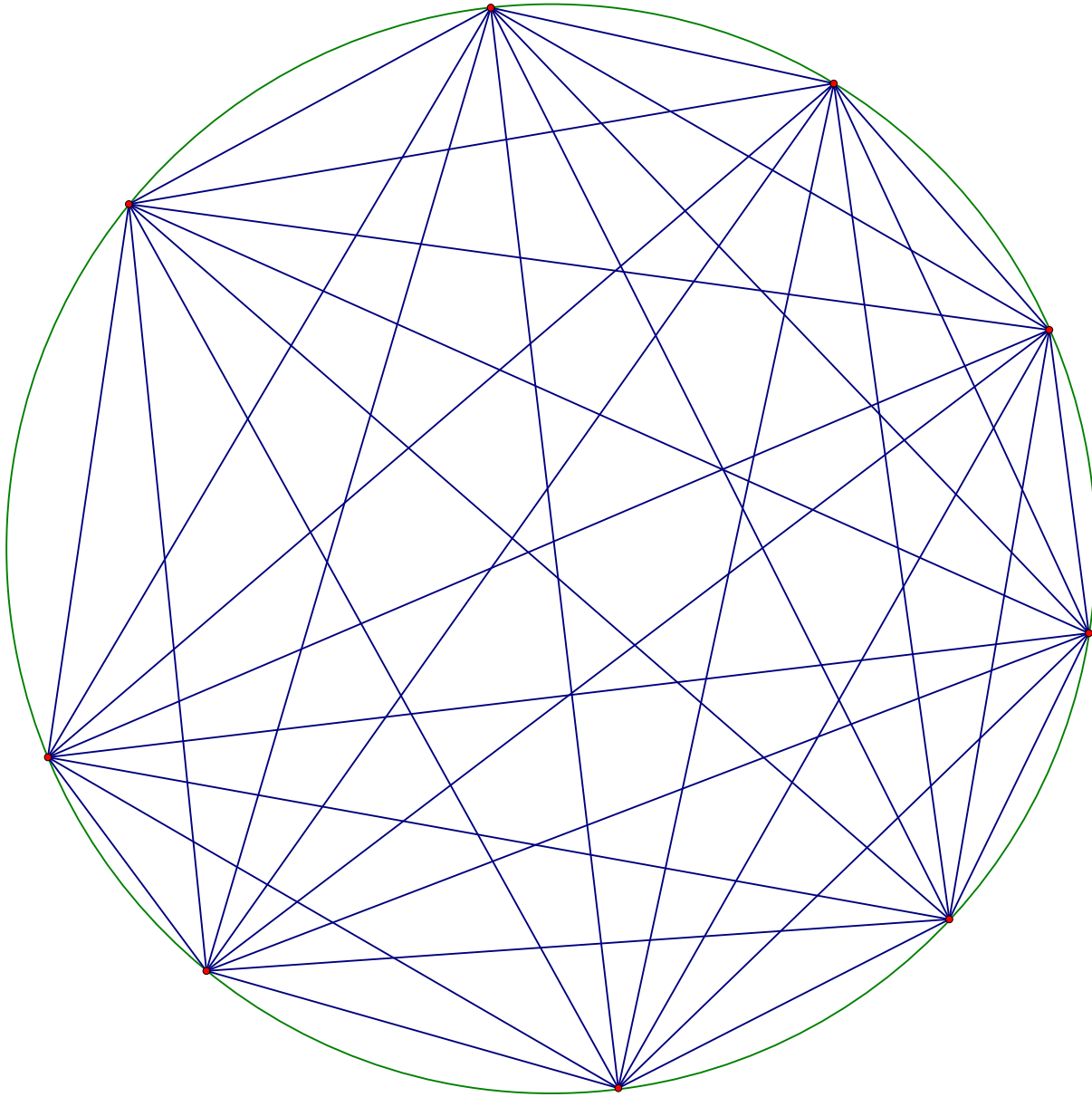
A Hanging Chain - Parabola or Not a Parabola

Materials Needed: cm graph paper, push pins, and a reasonable heavy yarn or thin rope.

1. Each team of students should either use some or all of a chain or necklace they are wearing, or to cut a piece of thin rope a length somewhere between 15 and 30 cm long.

2. Turning the cm graph paper into the horizontal position, mark the origin (0,0) in the center and at the bottom of the page. Use push pins to anchor the ends of the hanging chain so that the lowest point of this symmetrical curve is at the origin, and the line of symmetry is the *y*-axis. It is important to try and have the push pins holding the ends of the chain at integer coordinate points (*lattice points*) and the vertex at the origin, so only use as much of your chain or rope to make this possible. Once the chain has been anchored in place, do a reasonably good job of tracing the curve made by the chain onto the graph paper. This will be at least a 2-person job, and pushing the pins into a bulletin board (if handy) works very nicely.

3. The diagram below is an example of what such a curve would look like if it were anchored at (4,7) and (−4,7).

Push pin (-4,7)　　　Push pin (4,7)
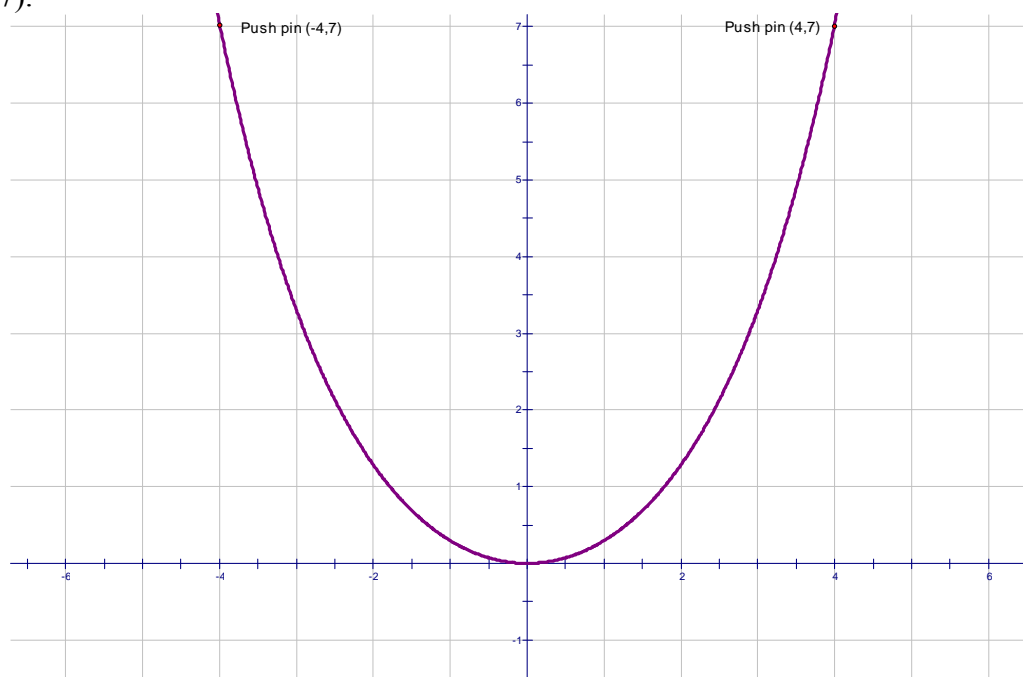
4. Extract data from the curve above to test out the hypothesis that the curve is a parabola. Carefully show your work in the space below.

5. Repeat step 4 above by using the curve you have traced out from your hanging chain. Once again, show your work in the space below.

6. Hopefully you have discovered that indeed this hanging chain curve is NOT a parabola. Instead, it is called a *catenary*, which is derived from the Latin word for chain. if you thought it was a parabola, you are in good company. Galileo himself thought this was the case until he was proven wrong by Jungius in 1669. Two famous mathematicians named Leibnitz and Bernoulli were among those who discovered the equation for the catenary in 1691. The equation if the vertex is at the origin is given by $y = \dfrac{a}{2}\left(e^{\left(\frac{x}{a}\right)} + e^{\left(-\frac{x}{a}\right)} \right) - a$. ($e \approx 2.718$ is a constant, and the parameter **a** controls the shape.)

7. Use the equation given above and your calculator to find the *a* value for both the given curve on the previous page as well as the curve produced by your hanging chain. Graph the curve on your calculator and look at a table of values to help assess how good a job you did.

Curve from previous page –

Your curve -

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A Model For Infectious Disease Spread

Materials needed: A box with a top of reasonable size (5x7 thru 11x14) for every three students, approximately one 2lb bag of M&M's for every two groups, graph paper, calculator.

The Problem: A small number of people have an infection that will potentially spread among the entire population. What will the graph look like that number of infected people is plotted versus time. Assume that infected people do not become cured, and they remain contagious throughout.

1. The M&M's are going to represent the people in the population. Let's distinguish healthy M&M's from infected M&M's by using dark colors to represent healthy and light colors (yellow) to represent sick. Place 200 "healthy" M&M's into your box.. Replace **three** "healthy" M&M's with three sick M&M's.

2. Each time period (say a day), will be modeled by shaking up the box and seeing what happens with the sick M&M's. Any sick M&M touching a healthy M&M will transfer the infectious disease. Put the top on the box and shake it up. Place the box back on the table and remove the top. Carefully count the number of healthy M&M's that are being touched by infected M&M's. After you and your partners have completed the count, remove the correct number of healthy M&M's that were just infected, and replace them with yellow or sick M&M's. Record the data in the table below, and also on the master table on the blackboard. It is a good idea for adjacent groups to choose different colors to represent sick M&M's, and it is also a good idea to let two different colors represent sick M&M's as their numbers grow larger.

| # of Days since beginning	0	1	2	3	4	5	6	7	8	9	10
# of Healthy M&M's	197										
# of infected M&M's	3										
Percentage of infected M&M's expressed as a decimal	.015										

3. Repeat the shaking up of the M&M population in the box, the counting of newly infected M&M's, and the recording of the data. Do this until at least 95% of the original population is infected. You may need to extend the above table

4. Once all the data has been recorded, enter the 1st and 4th rows of data into the calculator into L1 and L2 respectively, and show a scatter plot of L1 vs L2. Also, make a good scatter plot on graph paper of the same data using appropriate scaled coordinate axis.

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5. Discuss among your team members some of the important features of your graph, and whether or not any of the mathematical functions we have studied would be appropriate for modeling this graph. Also discuss what may account for any differences in the data among the groups performing this experiment. Use the space below to write any observations.

6. As you probably have discovered when thinking about this problem, none of the familiar functions we have studied seem appropriate for modeling this situation. While an exponential growth model is tempting because the increase of those infected is proportional to the number already infected, this does not take the diminishing opportunity into account. The more M&M's infected, the **fewer** there are to infect in the next iteration. A simple recursive quadratic model for this may be just what we need. This assumes that the number who become infected in any day is proportional to the number already infected, and **also** is proportional to the number not infected. This recursive model is described by the following equation. $\begin{cases} P_n = P_{n-1} + kP_{n-1}(1 - P_{n-1}) \\ P_0 = starting\ \%\ of\ infected \end{cases}$. P_n stands for the percentage of infected after n days, P_{n-1} is the number infected after $n-1$ days, k is the unknown proportionality constant, and P_0 is the starting percentage of the population that is infected.

7. For instance, if the total population of M&M's was 200 and we started with 3 of those 200 as being infected, so $P_0 = .015$. If you use the MODE button on the calculator to put into SEQUENCE mode, then recursive relationships like the one we are using for this model can be entered into the calculator. The instructor will help lead the class through one example on the board. The goal will be to find an appropriate proportionality constant k that will lead to the mathematical model being a reasonable close approximation to the actual data you collected.

8. Once the class has reviewed using SEQUENCE mode on the calculator and finding a k value for the classroom example, repeat the process for your own data. Try and come as close as possible to modeling the data you collected.

9. Try and think of an extension or a change to this problem to make it slightly different? How would you change the M&M game to take them into account? How would you change the mathematical model to take them into account?

The Slope of an Exponential Function at its *y*- intercept

Materials needed: Graph paper, straight edge, and calculator.

1. The goal is to see if there is a way to predict the slope of the tangent line for an exponential curve of the form $y = b^x$ at its *y*-intercept of $(0,1)$. The example at right shows the curve $y = (1.5)^x$. We are going to estimate the slope of the tangent line at $(0,1)$ two different ways. First, use a straight edge to draw an approximate tangent line at $(0,1)$ by just eyeballing it. Calculate or measure the slope of your line and record the slope in the table below. Method two is to use your calculator to determine the *y*-coordinate of the point on the curve whose x-coordinate is 0.01. Then use a $\Delta y / \Delta x$ calculation with $(0,1)$ to estimate the slope of the tangent line. Also, record your slope estimates in the table below.
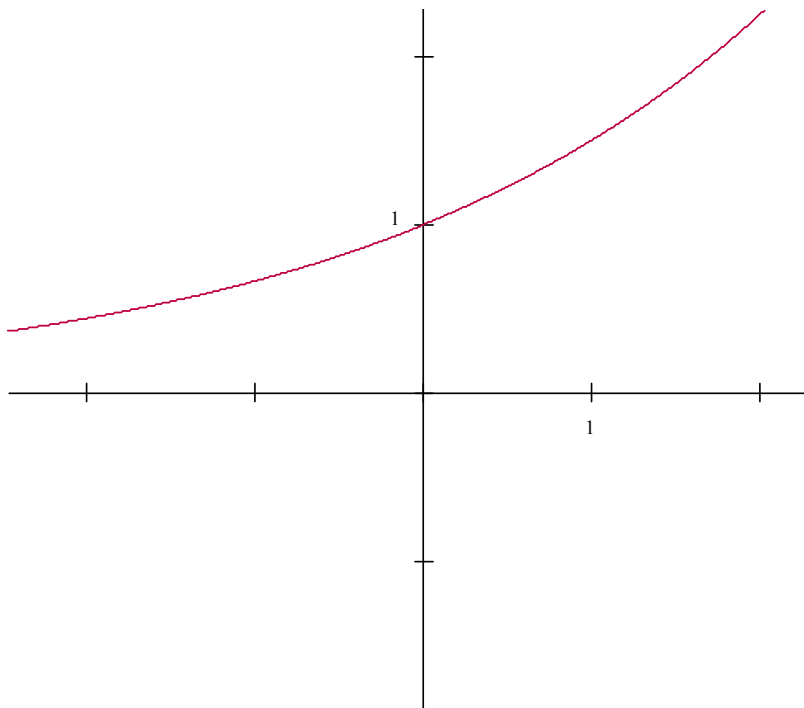
| Base *b* in $y = b^x$	1.5	0.3	0.6	0.9	1.2	2	2.5	3	3.5	4	5
Estimate by eyeballing with straight edge											
Estimate by $\Delta y / \Delta x$ calculation using a point nearby											

2. Both of your estimates above should have been somewhere around 0.4. If that is the case, you are ready to go to the instructor and be assigned one of the other *b* values in the table above. If not, go back and check your work before proceeding.

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3. Once you have been assigned another exponential curve. Your first job is to make a reasonably accurate sketch of the curve for $-1 \le x \le 1$. Use a large scale on your graph paper, and use the calculator to determine the coordinates of the points to plot in order to sketch your curve.

4. Once you have your curve drawn, repeat the slope approximation procedures described in step #1 and record your results in your data table, and also on the common data table on the board.

5. Once the data table has been completed by the class, enter the data into the calculator with the first row going to L1 and the third row going to L2. Do a scatter-plot of L1 versus L2 on the calculator and also on a piece of graph paper.

6. Looking at the graph, is there a familiar function that you are reminded of? If you said a logarithm function, you are exactly right. If this data did come from a logarithm function of the form $y = \log_c x$, the unknown is the base of the logarithm c. However, since we have lots of data points, we can choose any of them to substitute in for x and y and then solve for c. Recall that $\log_c x = \dfrac{\log x}{\log c}$ which is helpful in solving this equation. Use the data point you calculated and solve for c. Compare with other groups around you.

7. Enter the function $y = \log_c x$ into the calculator and graph in order to see if it indeed goes thru the data points from the scatter plot. To enter this into the calculator, use $y = \dfrac{\log x}{\log c}$.

8. The c value that you have approximated should be reasonably close to 2.7. This is a close approximation to the base of the natural logarithm LN(x). The constant e is in honor of the famous mathematician Leonhard Euler (1707-1783). You will notice on your calculator that the LN button is just below the LOG button. Just like the exponential function 10^x is the inverse function of LOG(x), e^x is the inverse function of LN(x). Enter $y = \ln(x)$ into the calculator and graph. Given the discussion you should not be surprised by what you see.

9. If you go back to the original data table, you will notice that for the base values of 2.5 and 3 the slope of the exponential function went from just under 1 to just over 1. What is the base of the exponential function whose slope equals one at its y-intercept?

10. Hopefully your answer to the above question was e.

Distinguishing Graphs

Problem: The three graphs depicted in this lab look somewhat similar. However, one of them is part of a quadratic function of the form $y = a(x+h)^2$, another is an exponential function of the form $y = b^x$, and the third is a part of a circle. All the graphs pass thru the point (0,1). Use whatever means you can think of to discern which graph is which. You may want to transfer them by tracing onto graph paper. Whatever you think will work. After you have which graph is which, determine the equation for that curve.
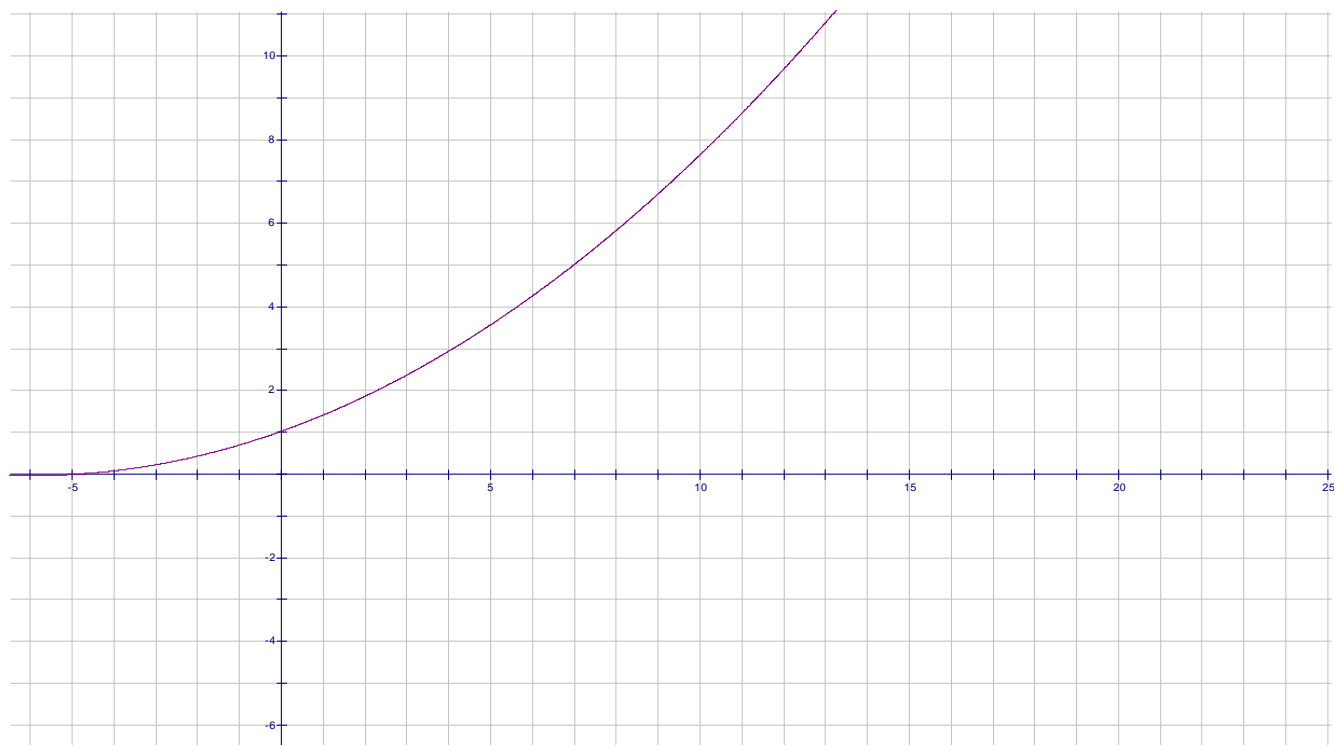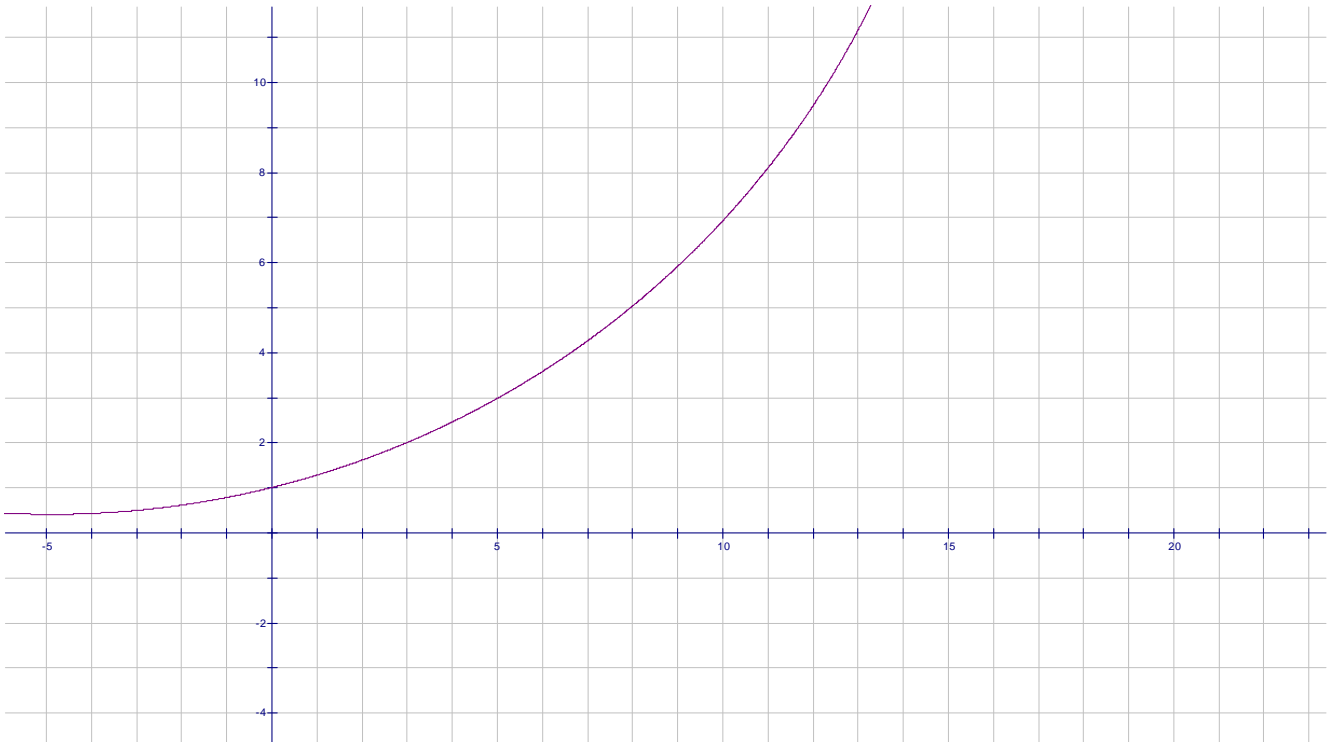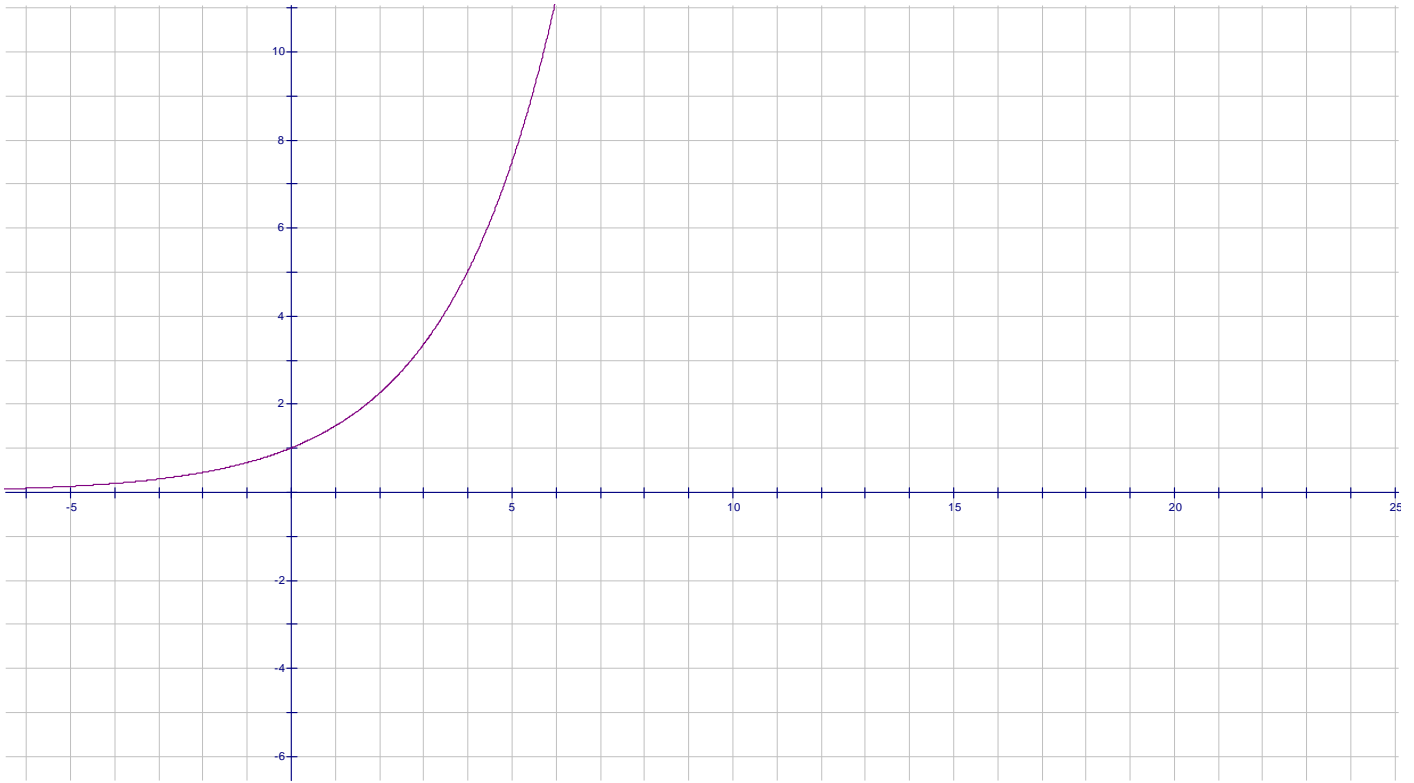
Rectangle Area Optimization Investigation

Materials needed are cardboard stock cm graph paper, scissors, calculator, and ruler.

1. A line is drawn with the equation $4x+3y=18$. The inscribed rectangle in the first quadrant *ABCD* is the object of this investigation. *A* is at the origin, *B* is on the *x*-axis, *C* is on the line, and *D* is on the *y*-axis. Specifically, we want to find the dimensions and the area of the maximum rectangle that can be inscribed as described. Start with a piece of cm graph paper in the horizontal position, locate the origin in the lower left hand corner of paper, let two cm spaces on the graph paper equal one unit, and graph the line $4x+3y=18$.
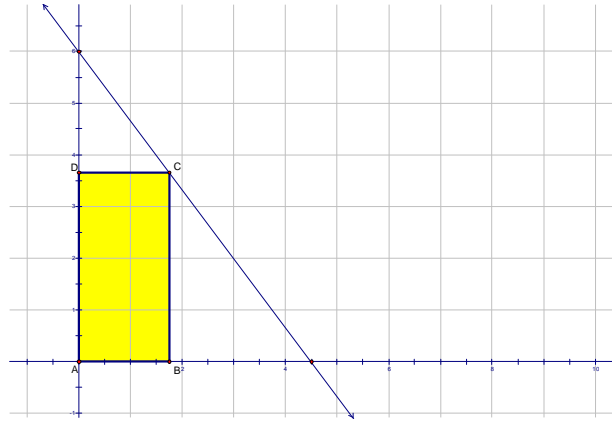
2. Setting the length of *AB* to a specific value between 0 and the *x*-intercept of the line determines the coordinates of points *B* and *C*, the dimensions of the rectangle, and thus its area. Let *AB* = 1. This means the coordinates of point *B* equals (1,0). Draw the inscribed rectangle on your graph paper, then in the space below, determine the coordinates of point *C* and the area of this rectangle.

3. When you have finished step #2 above, and correctly determined the coordinates of *C* to be (1,14/3), and the correct area to be 14/3 $units^2$, then it is time to go to the instructor and be assigned one of the *AB* lengths in the table below. Once you are assigned a new value for *AB*, you will repeat step #2 using this value. Thus, you will draw the inscribed rectangle with this new *AB* dimension, determine the coordinates of *B* and *C*, and determine the area of rectangle *ABCD*. When finished, write the information in the table below and in the table on the board. Also, cut out the rectangle and write the area in the center of the rectangle.

Length of AB	Coordinates of B	Coordinates of C	Area of rectangle
1	(1,0)	(1,14/3)	14/3
0.5			
1.5			
2			
2.5			
3			
3.5			
4			

4. Copy the information from the entire class into your chart on the previous page, and also enter the data into your calculator. Do a scatter plot of the data in your calculator.

5. In the table below that looks exactly like the one on the previous page, the variable x is now in the first column for length of AB. Determine the entries in the other columns that will all be in terms of x.

Length of AB	Coordinates of B	Coordinates of C	Area of rectangle
x			

6. The expression for area in the last column gives us the area as a function of x. Enter this into the calculator in y_1. Note: Once you have the function entered, look at a table of values for this function, and also graph the function. The window currently in the calculator from your scatter plot in #4 should be perfect to see the graph. Find the minimum value of the function, and also the x-value that produces that minimum. What observations about the table and the graph can you make below? Is the function describing area a familiar function?

7. When you have reached this point, go to the instructor and obtain a new point linear equation from the table on the next page to replace $4x + 3y = 18$. Once you have been given this new linear equation, you will repeat steps #5 and #6 above to determine the maximum area rectangle inscribed in the first quadrant for that particular line, and the x value that yields the maximum. Enter your new information in your table on the next page, and on the board.

Equation of line	x-intercept of line	y-intercept of line	Coordinates of B when area is maximum	Coordinates of C when area is maximum	Maximum Area of Rectangle
$4x + 3y = 18$					
$x + 3y = 9$					
$3x + 4y = 12$					
$2x + 5y = 20$					
$4x + y = 9$					
$3x + 5y = 18$					
$5x + 7y = 35$					
$x + 2y = 12$					

8. Looking at the results in the table above, write observations in the space below and try and prove any conjectures.

An Electronic Darts Simulation

Materials needed: TI-84Plus/Silver Ed., straight edge, compass

We shall break the class into four or five teams, as appropriate, and these teams will be working together throughout this investigation.

The plan: We will pose a question involving the probability of a particular event occurring and our job will be to try to simulate the occurrences that are sample points in this event on our TI graphing calculator. It is our goal that this simulation will lead us to a deeper understanding of the mathematics involved in computing the probability of the event described.

The event: Two random numbers are chosen, say *rand*1 and *rand*2, and we ask ourselves what would be the probability that $rand1^2 + rand2^2 < 1$.

0]   The members of each team should discuss this question just posed and then declare an educated team guess about what this probability might be.

0.5]   Let's take a minute and examine how the calculator seems to generate "random" numbers. To this end, consider the following simple routine, entitled RANDY000, written for the TI-84:

```
: For(I,1,3)
: Disp rand
: Pause
: End
: Disp " SO TI "
: Disp " RANDOMIZES "
```

We should run this little program a few times. What do you see? What about the size of all of the numbers generated?

1]   Let's gather some data. Each team should record date for 20 sets of random number pairs, (*rand*1, *rand*2). The random number command can be found under MATH/PRB/1 on the TI-84. These ordered pairs, (*rand*1, *rand*2), will serve as the ***electronic darts*** referenced in the title of this **lab**.

We want to record the team results for a mini-simulation of 20 ordered pairs of random numbers, (*rand*1, *rand*2). Make sure that your team has a clear understanding of how you turned the raw data into your team estimation of the probability of the occurrence of **the event** described above.

2]   We now will pool all the raw data gathered by the teams and come up with a new estimate of the probability based upon a much larger sample. So let's record carefully our total number of ***nice*** e-darts and the total number of e-darts considered by all the teams taken together. And let us record our probability estimate this time.

We might want to think about how this estimate might be related to the particular estimates of the various teams?

3]   We need more data concerning the number of nice edarts out of a total number of edarts considered but the repetitive nature of this work is probably getting to be a bit tedious! So let's write a short routine for the calculator to get our technology to help us out here. The following is a shell form of such a routine (to compute and count the nice darts) written in folksy language only:

> : Set a counter, for nice edarts at 0 (0 → K)
> : Set the number of trials, N
> : For pass # v, (v ≤ N)
> :   rand → R
> :   rand → S
> :   If $R^2 + S^2 < 1$
> :   then count that edart (K+1 → K)
> : Next value for v (loop back to For statement)
> : Display K

4]   Let's get this routine into our TI. Some of you may want to try to write up a routine such as this one for your TI or we can share electronically the following program that does this work for us:

> PROGRAM:EDARTSUM
> : 0 → K
> : Disp " HOW MANY TRIALS "
> : Input N
> : Fix 9
> : For(V,1,N)
> : rand → R
> : rand → S
> : If $R^2 + S^2 < 1$: Then
> : K+1 → K
> : End
> : End
> : Disp "NICE EDART SUM"
> : Disp K

We probably should let the programming gurus in the class explain why two End statements are needed in the above routine.

5]   Now we want each group to use the above routine to run 20 simulations to gather more data electronically. Keep careful count of the successful trials (nice edarts) and the total trials (edarts). You should have twenty pairs of numbers: nice edarts and total edarts. During each simulation (run of the routine, EDARTSUM), you can select any number for how many trials you want the machine to compute. But don't get too carried away or you might melt (lock-up) your calculator. 500 or 600 trials or thereabouts seems to be a reasonable number for a given simulation run.

Count up with care the team totals for all the 20 runs taken together. Record your team totals as total nice edarts computed and the grand total of edarts considered.

6]   Let's now combine all of the group data.

Total Successful trials (nice edarts)   _____

Total Trials considered (total edarts)   _____

Let's compute our new (large sample) probability estimate and compare it to our earlier guestimates and estimates. Hmmm…..

7]   Team summaries and class conclusions

What should we make of all this? Pictures and careful diagrams will help out here.

1)   Draw a careful picture of the portion of the coordinate plane which could be thought of as the target for our electronic darts (edarts). Where in the coordinate plane would all such edarts have to land? And why?
2)   How many edarts might there be that could land in this target region theoretically? Hmmm…
3)   Plot as many nice edarts as you can in your team diagram. The more the better but you should aim for a careful depiction of at least 50 nice edarts.
        What do you see? Is their any discernable pattern?
4)   Let us now pool all of our edart data and form a master picture of the situation on the board for the class to share. Each team should plot at least 40 nice edarts in the master picture.

8]   Conclusions and connections

9]   Run the program FULLMONT

This program is available electronically for you to load onto your calculator electronically. Just ask and you shall receive.

Cooling Coffee -Comparing and contrasting two possible models

For this LAB we shall break up into lab teams (three or four participants per team) and work together to investigate in class the issues raised.

The Problem

Coffee has been poured into your teacher's coffee mug and it is of interest to be able to predict the temperature, *H*, of the coffee in the cup as it cools over time, (in minutes). For this LAB, we assume that the temperature, *H*, is being measured in degrees Celsius.

So let us agree to start our clock of observation and prediction at time *t* = 0, when the coffee is first poured into the mug.

Model 1: An Explicit Model

Here the temperature, *H*, of the coffee at time *t* minutes is given by the following explicit formula for *H(t)*:

$$H(t) = 20 + 60 \Box 0.95^t$$

Each lab team should use this model to answer some specific questions regarding the temperature of the coffee at particular times. Make sure you record your basic analysis as well as your results so you can explain your approach to the full class.

Some questions follow:

1] What was the temperature of the coffee when it was first poured into the mug?

2] What would the temperature be ten minutes later? One hour later? Two hours later? Three hours later?

3] How many minutes would have to pass before the temperature of the coffee has cooled to 65° Celsius?

4] How many minutes would have to pass before the temperature of the coffee has cooled to 18° Celsius?

5] Do you have any conjectures concerning the ambient temperature in the room containing the mug of cooling coffee? Explain, please.

6] Discuss with your team members how such as model might have been derived. What information would have to have been documented and recorded to come up with a model for this situation?

Model 2: A Recursive Model

Here we describe the temperature of the coffee as being given by the following formula group, where H_n represents the temperature of the coffee, in degrees Celsius, n minutes after the coffee has been poured:

         i) $H_0 = 80^o$,   here $n = 0$, giving the initial temperature

             and

         ii)   for all $n \geq 1$, $H_n = 0.95 \cdot H_{n-1} + 1$, the **_recursive relationship_** defining
             H_n in terms of the previous temperature, H_{n-1}

Let us now use this recursive model to answer the very same specific questions regarding the temperature of the coffee at particular times. As before, you should be careful to record your basic analysis as well as your results so you can explain your approach to the full class.

Note on notation: It is traditional to use the variable, t, to represent the time in minutes for **model 1** (explicit/closed formula) and to use the variable, n, to represent the time in minutes in the recursive **model 2**. Do you have thoughts on why these conventions might have evolved?

Now you should examine carefully your answers to the questions that you computed using each model. What seems to be the same? What seems to be different?

What does your team see as the strengths and the weaknesses of each model? Be prepared to support your claims.

Suppose you were asked to predict the exact temperature of the coffee twelve hours after the coffee had been poured? What would you expect it to be? Does your **model 1** prediction seem to be in concert with your result using **model 2**?

Your answers to the many questions asked up to this point should suggest that these models might just be different ways of describing the exact same situation. We can use our technology to test this hypothesis in greater depth.

More Lab Team Work:

It might well prove interesting and perhaps enlightening to try to get a display on our calculator of the values of $H(t)$ as compared to those of H_n. We would then need to agree upon a **MODE** for our calculator that could do both outputs at once! And we would then need to decide upon a common input variable. Teams should be prepared to explain why we would have to decide between t or n, since it is clear that we can't use both at the same time.

See if your Lab team can come up proper mode and window settings for your TI, with u(n) being the explicit rule given in model 1 and v(n) the recursive rules given in Model 2, to produce a display on your calculator similar to the following:

n	u(n)	v(n)
0.00000	80.000	80.000
1.0000	77.000	77.000
2.0000	74.150	74.150
3.0000	71.443	71.443
4.0000	68.870	68.870
5.0000	66.427	66.427
6.0000	64.106	64.106

n=0

Follow-up question:

Do you believe that these two models are describing exactly the same situation? Be prepared to explain to the group how your lab team investigated this question. And be prepared to support your claims?

Could you present a formal **proof** of your claims here?

Hint *for ambitious proof-hungry teams:* It might prove profitable to try to use some precalculus mathematics to turn **model 2** (recursive model) into **model 1**. Or you can appeal directly to *Newton's Law of Cooling* for a reasonably quick confirmation of the claim. But any (logically sound) method you hit upon is fine.

Exploring Some Special Spirals

This **lab** will use technology to help us develop some understanding of a special class of spirals described in polar coordinates.

We will split up into **lab teams** of three or four participants per team and work together within teams to try to develop and extend our understanding. We will need **TI-83 (Plus)** or **TI-84 (Plus)** graphing calculators and about 10 sheets of polar graph paper per team for this lab.

Each lab team should take a few minutes to be sure that you are all on board about how polar coordinates and rectangular coordinates are related and how polar curves are plotted/graphed.

Each team member should come up with twenty points in polar coordinates that must be on the polar curve $r = \cos(\theta)$. Plot all the team points carefully on a common team sheet of polar graph paper and see what you can figure out. Add a few more points to your graph. ***Hmmm…***

Come up with ten to twenty points per team member in polar coordinates that must be on the polar curve $r = \sin(\theta)$. Plot all these team points carefully on a common sheet of polar graph paper and see what you can figure out. Add a few more points to your graph. ***Hmmm…***

How do these two polar curves seem to relate to one another geometrically/visually?

Can you substantiate your claims with some mathematics? Each team should try to do so.

Things get more interesting if we take a simple exponential expression of the form b^x and use this concept to examine the graphs of polar curves of the form $r = b^\theta$, where b is some positive real number representing the base of our exponential expression ($b \neq 1$) and θ is measured in degrees (for this lab).

Each team should take the time to examine the following polar curve of the above form with some care: $r = 3^{\frac{\theta}{360}}$. What is the base, b, in reality here? Note that it is *not* 3!

Each lab team member should plot at least fifteen points and all these points should be plotted carefully on a common polar diagram.

Could $\theta - values$ be very large? How large? Could such $\theta - values$ be negative? How would such polar points be plotted? Be prepared to explain your team findings to the group as a whole.

The class should select a participant who has already demonstrated exceptional ability as a graphics artist and that person should try to represent the polar curve under investigation, $r = 3^{\frac{\theta}{360}}$, in fully (enlarged) glory on the board for all to see. Does it look like a ***spiral*** to you?
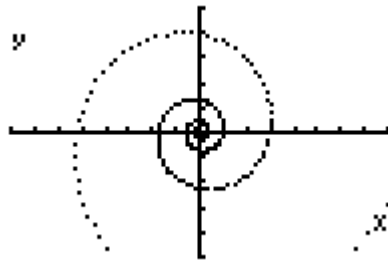
Lab teams should take some time to try to find proper **window** settings for your **TI-83/4** to replicate as accurately as you can the above picture, taken directly from the screen of a **TI-83 Plus**, using the given polar equation $r = 3^{\frac{\theta}{360}}$.

Let us now add a *ray*, emanating from the polar origin, to our graph. For practice, say we choose the ray, $\theta = 120^0$, where r is restricted to non-negative values only here. Make sure you understand and can explain to your classmates why the graph of the ray comes out looking as it does.

This special ray must meet the spiral in the second quadrant at a point on the polar spiral, $r = 3^{\frac{\theta}{360}}$. Each lab team should examine the following questions with care:

What are the precise coordinates of this point, P, on the spiral in polar coordinates: $(r ; \theta)$?

And what would be the coordinates of this same point, P, expressed in rectangular coordinates?

All the teams should compare results here and make sure that they are in full agreement on their results up to this point.

We now need to approximate the slope of the tangent line to the spiral at the same point, P, on the curve determined by the above ray. There are various ways to do this. Explore this problem with your teammates and try to come up with an accurate value approximating the slope of this tangent line at P.

Lets all compare our team results. If we disagree, then let's find out why and resolve any conflicting results.

Now each team should try to perfect an algorithm for determining the angle between the spiral (determined by the tangent line to the curve at P) and the ray selected above at P.

The situation now looks something like the following for $\theta = 120^0$:
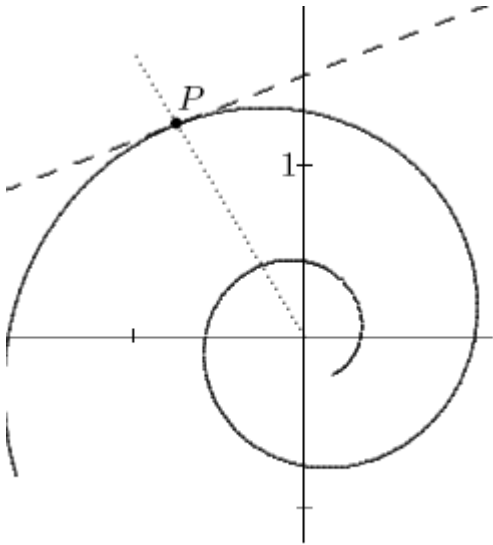
Let's now compare team results. How do our angular measures compare? Let us compare the team approaches to this calculation and make sure we understood what the various teams did and why. Do our results agree?

We now need to follow the same procedure again for determining a precise value approximating the angle between the spiral at a new point **P1** (determined by its tangent line there) and the ray **OP1** , from the origin, **O**, through the new point **P1**.

 This time let us use the ray determined by $\theta = 30^0$ (r still restricted to non-negative values here).

Each team should come up with a value for the angle between the spiral and the ray **OP1** this time. You should make sure to record your result and the method you used to determine it. Be prepared to explain what you did to your classmates.

Let's compare team results this time around. Do we agree? If not, then let's correct any errors made so that we are in full agreement with our results to date.

Now each team will be assigned two new rays emanating from the origin. Each assigned ray will be one of the two rays assigned to at least one other team. So the angle between the spiral and each ray will have been computed by at least one other team. Thus we can have some basis for checking how we are doing.

Results should be compared with the appropriate team for each ray assigned. Make sure that the teams make adjustments as needed so that they agree on all the various computed angular values.

What do we see?

Each team should now allow some time to explore the following sibling spirals with some care:

(i) $r = 2^{\frac{\theta}{360}}$ (ii) $r = 4^{\frac{\theta}{360}}$ (iii) $r = 5^{\frac{\theta}{360}}$ and (iv) $r = \dfrac{1}{2}^{\frac{\theta}{360}}$

In each case teams should be prepared to consider the following questions as fully as possible:

      1] Does each polar curve seem to have a constant angle between any given polar ray and the tangent line to the curve where the ray meets the curve?

      2] If your team answer is no, then be prepared to present examples in support of that position. But if your team found that each of the polar curves seemed to have such a common angle (as defined above), then what was the measure of that constant angle in each case (accurate to three decimal places or so)?

We seem to be dealing with a remarkable family of **special spirals** here. We can, in fact, use some technology to help us gain deeper insight without having to spend much of our time involved in fairly tedious computations. We need only turn to our TI graphing and programmable calculators for some help in this regard. Then we can explore several different members of this special family of polar curves relatively quickly and easily.

Each team might want to take a few minutes to think about how one might write a routine for the calculator to do our angular computations for us. What would the machine need to know in order to make the angular computations?

Some issues to be considered might include: proper MODES, values selected for the base, b, and for the input variable, θ, computational accuracy selected, etc…

 Some teams might want to try their hand at writing a short program for the TI – 83/4.

We will then come together to attack this project as a class.

Consider the following calculator program, called **POLSPIAN**:

```
Disp "POLAR MODE"
Disp "SPI IN r1"
Disp "INPUT"
Disp "RADIAL ANG"
Disp "IN DEGS"
Prompt T
r1(T)*cos(T) →  A
r1(T)*sin(T) → B
Disp "RADVEC"
Disp A, B
Pause
Disp "DELTA-T"
Input U
T+U → T
r1(T)*cos(T) → C
r1(T)*sin(T) → D
C-A → R
D-B→ S
```

Disp "TANVEC"
Disp R, S
Pause
Disp "SPIANG IS"
Disp "APPROX"
Disp " "
Disp " "
Disp cos $^{-1}$ ((A*R+B*S)/($\sqrt{}$ (A^2+B^2)*$\sqrt{}$ (R^2+S^2)))
Disp "HMMM..."

This program can easily be sent to interested participants electronically. Or participants might prefer to write their own routines, quite possibly improving dramatically on the one displayed above.

The following are two separate runs of this program, **POLSPIAN**, one for $\theta = 120^0$ and another for $\theta = 45^0$.

```
SPIANG IS
APPROX


     80.08211140
HMMM...
           Done

SPIANG IS
APPROX


     80.08217143
HMMM...
           Done
```

But you may well remain unconvinced.   Perhaps the writer just input the same information in two separate runs of the program. We will now run the program in class together to try to alleviate your fears and maybe even convince some of you that the screens above really were obtained using the claimed data sets ($\theta = 120^0$ and $\theta = 45^0$).

Nonetheless, the two results do seem to be very close. So what can we conclude? What have we actually *proven*?   What could we prove here?

And what is that ***strange formula*** used in the program to compute the angle between the specified radial and the spiral (local tangent line) there?

Have you seen this formula, which seems to involve vector concepts, before?   Could you derive it?

Let's begin wrapping up this lab by letting each lab team work on trying to derive the formula used for computing the angle in the program just described.

What do you think such spirals are called? And why?

Have you heard of spirals called *logarithmic spirals*? How do they relate to all this? And why?

## *Extra for experts but for another day:*

1]   Could one find a base, b, for such a spiral such that the constant angle between any radial vector and the tangent line at the determined point on the spiral would come out to be exactly 90^0? Hmmm…

2]   Could one determine a base, b, such that the constant angle between any radial vector and the tangent line at the determined point on the spiral would be exactly 80^0.

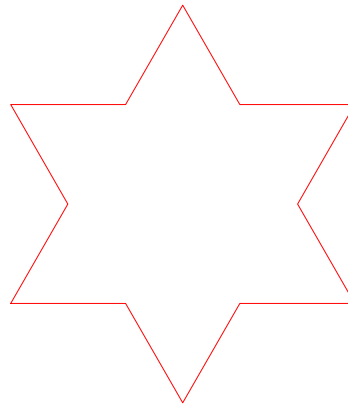3]   Could one determine a base, b, such that the constant angle between any radial vector and the tangent line at the determined point on the spiral would be greater than 90^0? If so, then how?

4]   You make up an interesting question expanding on these ideas. Then try to answer it.
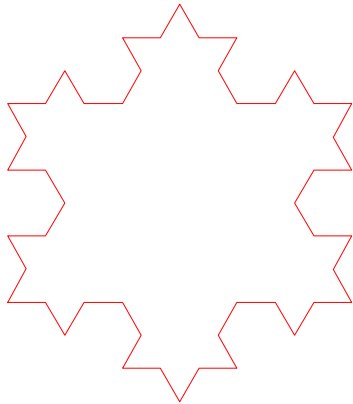
The Making of a Snowflake
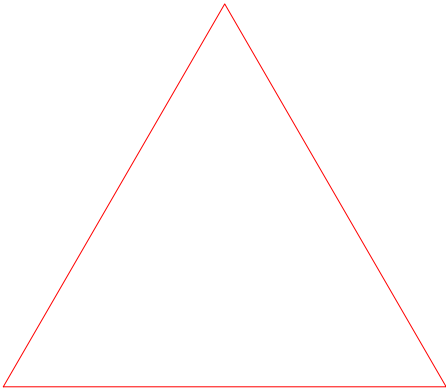
Stage 1

Stage 2

Stage 3

The Koch snowflake, is pictured in its first three stages above. The process begins with an equilateral triangle at Stage 1. At each subsequent stage, each edge of the figure is divided into thirds and on the middle third, an equilateral triangle is built. Try to sketch the top part of stage 4.

1. The number of sides in each figure forms a sequence. Write down the first few terms of the sequence. Write a function that defines this sequence. Is the sequence geometric? What is its common multiplier or ratio?

2. The lengths of each individual side also form a sequence. If the length of each side of the triangle in Stage 1 is 1, write the first few terms of this sequence and then write a function that defines this sequence.

3. From your work in problems 1 and 2, form a <u>sequence of the perimeters</u> of the figures. Is this sequence geometric? Does this sequence converge to a sum? Give a reason for your answer.

4. Now let's look at the area of the snowflake. Assume that the area of the first equilateral triangle is 1. Explain why the part of the original area <u>that is added on</u> in Stage 2 can be written as

$$3\left(\frac{1}{9}\right)$$

How many little triangles are added on in Stage 3?   What part of the original triangle does each little triangle represent in Stage 3?   Write an expression for the <u>total area added</u> on in Stage 3.

Repeat these steps for Stage 4.

5. The additional areas added at each stage form a geometric sequence. Justify that they do. [Hint: Do not reduce the fraction to lowest terms.]

6. You can now form a series with the pattern

   1 + area added stage2 + area added stage 3 + … . Do so.

7. The series starting with the second term (i.e. leaving out the 1 for a moment) form a geometric series. Is it finite or infinite? Explain. What number does this infinite series approach? Now include the 1 and obtain a value for the total area covered.

8. Consider your answer to question 3 in conjunction with your answer to question 7. It appears that this snowflake has an infinite perimeter but encloses a finite area. In other words, you could not buy enough wire to insure that you could make any given stage of this snowflake, but you could buy enough paint to paint its interior no matter how large the snowflake got. Explain how this is possible or where we have made an error.

The Maine – NH Truck Investigation

The *xyz* shipping company has two offices, one in New Hampshire and one in Maine. Records show that each day, 70% of the trucks in the New Hampshire office go to places in New Hampshire and 30% are sent to locations in Maine. Similar records at the Maine office show that each week, 60% of their trucks deliver in Maine and 40% go to New Hampshire. In every case, once the delivery is made, the truck returns to the office IN THE STATE IN WHICH THE DELIVERY WAS MADE.

1. Divide your group in two teams, one representing New Hampshire and one representing Maine. Each group begins with 50 pieces of candy. The following action will be repeated each "week".

2. Those in charge of New Hampshire do the following: Count out 70% of the pieces of candy. They stay in New Hampshire. Move the remaining 30% of the Starbursts to Maine.

   Those in charge of Maine do the following: Count out 60% of the pieces of candy. They stay in Maine. Move remaining 40% of the candy to New Hampshire.

BEFORE YOU ACTUALLY BEGIN MOVING CANDY:

Make a prediction as to what you think will actually happen after several (many) moves.

3. Now begin and record your numbers below. (Round off numbers to the nearest whole number of trucks.)

Week	New Hampshire			Maine		
	Stay in NH(70%)	Go to Maine(30%)	Total in NH	Stay in Maine(60%)	Go to NH(40%)	Total in Maine
1						
2						
3						
4						
5						
6						

Describe any patterns or trends that you see evolving.

4. Would your pattern hold if we tried unequal starting numbers? Let's test it by beginning with 80 trucks in New Hampshire and 20 Trucks in Maine. Repeat the steps above and record your results.

Week	New Hampshire			Maine		
	Stay in NH(70%)	Go to Maine(30%	Total in NH	Stay in Maine(60%	Go to NH(40%)	Total in Maine
1						
2						
3						
4						
5						
6						

5. Comment on what you observe happening.

6. Let *N* represent the number of New Hampshire trucks and *M* represent the number of Maine trucks at the end of a week.

   a) What algebraic expression represents the number *N* of New Hampshire trucks at the end of the following week?

   b) What algebraic expression represents the number *M* of Maine trucks at the end of the following week?

7. Try and set up a 2 x 2 matrix containing the percents in the correct locations so that when you multiply it by the 1 x 2 matrix [*N, M*] you get another 1 x 2 matrix containing the expressions from 6*a* and 6*b* above. In other words, replace with question marks in the following equation with the correct percentages so the resulting matrix product yields the correct expressions for the number of trucks in New Hampshire and Maine respectively in the product matrix.

$$\begin{bmatrix} N & M \end{bmatrix}\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} \exp 6a & \exp 6b \end{bmatrix}$$

8. If you came up with the matrix $\begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$ as the *transition matrix* that you multiply times the row

vector $\begin{bmatrix} m & n \end{bmatrix}$, you are a superstar. Use this matrix and the initial truck distribution state of [50 50] to recreate the data in the first table, and then recreate the data in the second table by using an initial truck distribution state of [80 20].

9. Do you see a calculator approach to determining the distribution of trucks over many weeks? What would you do?

10. Determine a different matrix if the percentages were changed so that 80% of New Hampshire trucks stayed in NH (20% to Maine), while only 50% of Maine trucks stayed in Maine. Use this matrix to determine the long term distribution of trucks starting with an initial distribution of 60 in NH and 40 in Maine.

11. What if the initial distribution in #10 above was 90 trucks in NH and 10 in Maine?

12. Try and show through algebra why the distribution of trucks eventually reaches the equilibrium state indicated by the calculator solutions in #10 and #11.