Mathematics 6

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1. The graph of an equation $y = f(x)$ is a curve in the xy-plane. The graph of an equation $z = f(x, y)$, on the other hand, is a *surface* in xyz-space. A familiar example is the graph of $z = \sqrt{9 - x^2 - y^2}$, which is a hemisphere of radius 3. For what points (x, y) is this function (and the surface) defined? Find an equation for the plane that is tangent to the hemisphere at $(2, 1, 2)$.

2. The graph of the equation $z = 9 - x^2 - y^2$ is a surface called a *paraboloid*.
(a) For what points (x, y) is the surface defined?
(b) Why do you think the surface was named as it is?
(c) Through any point on the paraboloid passes a circle that lies entirely on the paraboloid. Explain. Could there be more than one circle through a single point?
(d) The plane that is tangent to the paraboloid at $(0, 0, 9)$ is parallel to the xy-plane. This should be evident. It should also be evident that the plane that is tangent to the paraboloid at $(1, 2, 4)$ is *not* parallel to the xy-plane. Can you think of a way to describe the "steepness" of this plane numerically?

3. Verify that area of the parallelogram defined by two vectors $[a, b]$ and $[c, d]$ is $|ad - bc|$. Explain the significance of the absolute-value signs in this formula. The expression $ad - bc$ is an example of a *determinant*. The sign of a determinant is an indication of *orientation*, meaning that it can be used to distinguish clockwise from counterclockwise. Explain.

4. Suppose that a differentiable curve is defined parametrically by $(x, y) = (f(t), g(t))$ for $a \le t \le b$. The area of the *sector* formed by the curve and the two radii joining the origin to the initial point $(f(a), g(a))$ and the terminal point $(f(b), g(b))$ is equal to the value of the integral

$$\frac{1}{2}\int_a^b \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\,dt.$$

(a) First, check this result by applying it to an original example (one that *no one else* will think of) that you already know how to find the area of.
(b) Next, prove the formula by viewing $x\,dy - y\,dx$ as a determinant.

5. A wheel of radius 1 in the xy-plane rolls counterclockwise without slipping around the outside of the unit circle $x^2 + y^2 = 1$. A spot on the rim of the wheel is initially at $(1, 0)$. The curve that is traced by the spot is called an *epicycloid*. Make a sketch.
(a) Does this curve repeat itself? Explain.
(b) After the wheel has rolled a quarter of the way around the circle, what are the coordinates of the spot?
(c) After the wheel has rolled a third of the way around the circle, what are the coordinates of the spot?
(d) After the wheel has rolled a distance t along the circle (think of t as being between 0 and 2π), what are the coordinates of the spot? (It helps to first find the center of the wheel and then use a vector.)

6. Let $T(x,y)$ be the temperature at point (x,y) on a rectangular plate (a modern stovetop, perhaps) defined by $a \le x \le b$ and $c \le y \le d$. If T is a non-constant function, then is natural to wonder how to describe rates of temperature change. For example, if the values* $T(9.0,12.0) = 240.0$ and $T(9.03,12.04) = 239.0$ are measured, then it is possible to calculate an approximate value at $(9.0,12.0)$ for the *directional derivative* of T in the direction defined by the *unit* vector $\mathbf{u} = [0.6, 0.8]$. Do so. The actual value of the derivative is usually denoted $D_{\mathbf{u}}T(9.0,12.0)$. If you wanted to calculate a more accurate value for $D_{\mathbf{u}}T(9.0,12.0)$, what data would you gather?

7. The easiest way to "gather data" is to have an explicit formula for $T(x,y)$, so suppose that the temperature of the rectangular plate is known to be

$$T(x,y) = \frac{6000}{(x-10)^2 + (y-10)^2 + 20}$$

at each of its points. Calculate a precise value for $D_{\mathbf{u}}T(9.0,12.0)$, and compare it with your previous estimate.

8. The value of a directional derivative is of course affected by the chosen direction, so it should not be surprising to discover that $D_{\mathbf{v}}T(9.0,12.0)$ is not the same as $D_{\mathbf{u}}T(9.0,12.0)$ when $\mathbf{v} = [0.6, -0.8]$. Verify this by calculating $D_{\mathbf{v}}T(9.0,12.0)$.

9. For any unit vector $\mathbf{u}$, how are the numbers $D_{\mathbf{u}}T(x,y)$ and $D_{-\mathbf{u}}T(x,y)$ related?

10. When $\mathbf{v} = [0.6, 0.8]$, the value of $D_{\mathbf{v}}T(14.0,7.0)$ is intriguing. Calculate this number and explain its significance.

11. Given a temperature function T, what are the *isotherms* of T? What is their relationship to directional derivatives of T?

12. Return to the current temperature example T, and find the unit vector $\mathbf{u}$ that makes the value of $D_{\mathbf{u}}T(14.0,7.0)$ as large as it can be. Explain your choice.

13. Return to the current temperature example T, and notice that $D_{\mathbf{v}}T(10.0,10.0)$ has a predictable value, no matter what vector $\mathbf{v}$ is chosen. Explain.

14. The directions $\mathbf{u} = [1,0]$ and $\mathbf{v} = [0,1]$ are special, and many notations are in common use for the corresponding directional derivatives of T, which are known as the *partial derivatives* of T. For example, it is customary to replace $D_{[\mathbf{1},\mathbf{0}]}T$ by $\dfrac{\partial T}{\partial x}$, or D_1T, or D_xT, or even T_x (leaving out the differential indicator "d" entirely!).
(a) What are other possible notations that mean the same thing as D_yT?
(b) Express the value of $D_xT(9.0,12.0)$ as a limit.

15. Given a formula $f(x,y)$ for a function of two variables, the significance of the partial derivatives f_x and f_y is that
(a) each can be calculated by the familiar, straightforward technique of differentiating f with respect to one variable, while treating the other variable as a constant;
(b) any other directional derivative $D_{\mathbf{u}}f$ can be calculated easily by just combining the components of vector $\mathbf{u}$ with the known values of f_x and f_y.

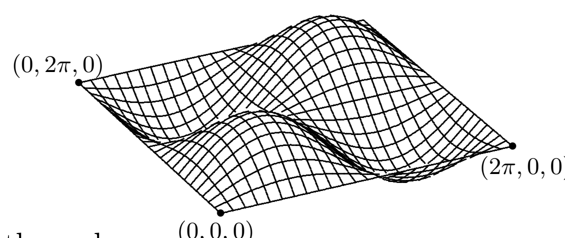
Consider the example $f(x,y) = x^2 + xy - 2y^2$, and the vector $\mathbf{u} = [0.6, -0.8]$. Investigate the preceding remarks by trying to calculate the values $f_x(1,3)$, $f_y(1,3)$, and $f_{\mathbf{u}}(1,3)$ as economically as you can.

16. Given the equation $4x^2 + 9y^2 + z^2 = 49$, whose graph is called an *ellipsoid*, calculate the partial derivatives z_x and z_y. Use these functions to find an equation for the plane that is tangent to the ellipsoid at $(1,2,3)$. Implicit methods work best here. Although this is *not* a surface of revolution, you should be able to draw a diagram.

17. (Continuation) Give several examples of vectors that are tangent to the ellipsoid at the point $(1,2,3)$.

18. Given a function f that is differentiable, one can form the vector $[f_x, f_y]$ at each point in the domain of f. Any such vector is called a *gradient* vector, and the function whose values are $[f_x, f_y]$ is called a *gradient field*. Suppose that $f(x,y) = 9 - x^2 - 2y^2$. Calculate the gradient vector for f at $(1,1)$. How is this vector related to the *level curve* $6 = f(x,y)$? Explain the terminology "level curve".

19. The surface $z = f(x,y)$, where $f(x,y) = \sin x \sin y$, and $0 \le x \le 2\pi$ and $0 \le y \le 2\pi$, is shown at right. Fifty curves have been traced on the surface, twenty-five in each of the coordinate directions. The two curves that go through $(0,0,0)$ are segments on the coordinate axes. Two of the curves go through $P = \left(\frac{2\pi}{3}, \frac{\pi}{3}, \frac{3}{4}\right)$.



(a) Find this point in the figure.
(b) Calculate the derivatives f_x and f_y at P.
(c) Write parametric descriptions for the curves through P (one in each coordinate direction).
(d) Find an equation for the plane tangent to the surface at P.
(e) Make a sketch that shows some level curves $z = k$ for this surface.
(f) What are the coordinates of the summit of the hill on which P is located? Suppose that a hiker wanted to follow the steepest possible path to the summit. Starting from P, in what direction should this hiker walk?
(g) Make a sketch that shows how this surface would look to an observer whose eye is positioned at $(20,\pi,0)$.

20. The point $(3,1)$ is on the ellipse $x^2+2y^2=11$. Find components for a vector that is perpendicular to the ellipse at $(3,1)$. Explain how the gradient concept can be applied to this question.

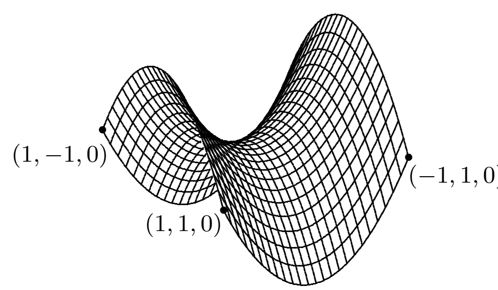
21. Given a surface $z=f(x,y)$, and a point $P=(a,b,c)$ on that surface, explain why
(a) the vector $[f_x(a,b), f_y(a,b)]$ is perpendicular to the curve $f(x,y)=c$;
(b) the vectors $[1,0,f_x(a,b)]$ and $[0,1,f_y(a,b)]$ are tangent to the surface at P;
(c) the vector $[f_x(a,b), f_y(a,b), -1]$ is perpendicular to the surface $z=f(x,y)$ at P.

22. Confirm that the space curve parametrized by $(x,y,z)=(4\cos t, 7\cos t, 13\sin t)$ lies entirely on the ellipsoid $16x^2+81y^2+25z^2=4225$. Confirm also that the curve goes through the point $P=(3.2,5.6,7.8)$ on the ellipsoid. Calculate **(a)** the velocity vector at P for this curve, and **(b)** a vector that is normal (perpendicular) to the ellipsoid at P. As a check on your calculations, verify that your answers are perpendicular.

23. Calculate the gradient field for the function $f(x,y)=x^y$. For what values of x and y does this make sense? By the way, the standard notation for the gradient is ∇f. That's right — an upside-down Δ means "gradient".

24. Evaluate the integral
$$\frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\,dt.$$
along the curve parametrized by $(x,y)=(t,\sin t)$. Is the answer what you expected?

25. The figure shows the surface $z=f(x,y)$, where $f(x,y)=x^2-y^2$, and $-1\le x\le 1$ and $-1\le y\le 1$. Fifty curves have been traced on the surface, twenty-five in each of the coordinate directions. This saddle-like surface is called a *hyperbolic paraboloid*, to distinguish it from the elliptical (or circular) paraboloids you have already encountered. It has some unusual features.



(a) What do all fifty curves have in common?
(b) Confirm that the line through $(1,1,0)$ and $(-1,-1,0)$ lies entirely on the surface.
(c) In addition to the line given, there is another line through the origin $(0,0,0)$ that lies entirely on the surface. Identify it.
(d) Find an equation for the plane that is tangent to the surface at $(-1,0.5,0.75)$.
(e) Make a sketch that shows some level curves for this surface.
(f) Explain the name "hyperbolic paraboloid".
(g) This surface contains *many* straight lines. In fact, for any point P on the surface, there are two lines on the surface that meet at P. Confirm that this is true for $(-1,0.5,0.75)$, by finding a direction vector $[a,b,c]$ for each line.

26. Suppose that $T(x,y)$ is a differentiable function and $\mathbf{u}$ is a unit vector. What does the equation $D_{\mathbf{u}}T = \mathbf{u}\bullet\nabla T$ mean? What does this equation tell you about the special case when $\mathbf{u}$ points in the same direction as ∇T?

27. Now that you have had some experience with directional derivatives, you can consider the following question: What does it mean for a function T to be *differentiable* at a point (a,b)? If you can, express your answer in limit notation.

28. If the function $T(x,y,z)$ describes the temperature of a substance at position (x,y,z), then what is the meaning of the vector $\left[\dfrac{\partial T}{\partial x}, \dfrac{\partial T}{\partial y}, \dfrac{\partial T}{\partial z}\right]$? What does the configuration of solutions to $T(x,y,z) = 23.6$ represent, and what could it be called?

29. *More fun with dot products.* Let $T(x,y) = 100 + 50\cos\pi x\cos\pi y$ for $-2 \le x \le 2$ and $-2 \le y \le 2$ describe the temperature distribution on a hot plate, and suppose that a bug is following the circular path $(x,y) = (\cos t, \sin t)$ on the plate.
(a) Sketch a system of isotherms for this function. Graph the bug's path on your diagram.
(b) When the bug passes over the point $(0.8, 0.6)$, at what rate is the temperature changing, in degrees per second? You can express your answer as a dot product.

30. When you first learned about tangent lines, you probably assumed that tangent lines must always stay on one side of the curve they are tangent to. Confronted by examples like $y = x^3 - 2x$ at $(1,-1)$, you then learned to regard one-sided behavior as a *local* phenomenon. The final blow to your intuition came when you encountered your first inflection point, where it is possible for a curve to lie locally on *both* sides of its tangent line.

Now that you have begun to explore planes tangent to surfaces, you should be ready to meet some more counter-intuitive examples.
(a) It is possible for a non-planar surface to be tangent to a plane, and to meet that plane along a line of intersection points. Give an example.
(b) It is possible for a non-planar surface to be tangent to a plane, and to meet that plane along *two* lines of intersection points, both of which go through the unique point of tangency. You have already seen an example — where?

31. Suppose that f is a differentiable function of x and y, and that (a,b) is a point at which ∇f is the zero vector. Is it necessarily true that $f(a,b)$ is either a local maximum or a local minimum? Explain.

32. You have seen that equations of the form $f(x,y) = k$ define *level curves* of function f. In the same way, equations of the form $g(x,y,z) = k$ define *level surfaces* of function g. The equation $x^2 + y^2 + z^2 = r^2$ is a familiar example. Invent another. Given such an equation, and a point (a,b,c) on one of the level surfaces, how could you quickly find a vector that is perpendicular to the surface at (a,b,c)? Illustrate by doing an example.

33. *from Math 210*: Find a vector that is perpendicular to the line $3x + 4y = 23$ at the point $(5, 2)$.

34. *from Math 330*: Find a vector that is perpendicular to the plane $2x + 3y + 6z = 53$ at the point $(4, 1, 7)$.

35. *from Math 420*: Find a vector that is perpendicular to the curve $x^3 + y^3 = 3xy + 3$ at the point $(2, 1)$.

36. *a new version of the Chain Rule*. If $z = T(x, y)$ defines a differentiable function, and if $(x, y) = (f(t), g(t))$ defines a differentiable path, the composite function defined by $h(t) = T(f(t), g(t))$ is also differentiable, and $h'(t) = T_x(f(t), g(t))f'(t) + T_y(f(t), g(t))g'(t)$. Using Leibniz notation and hiding the function names gives the equation a familiar look:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Justify this equation, and make up a new example that illustrates its use.

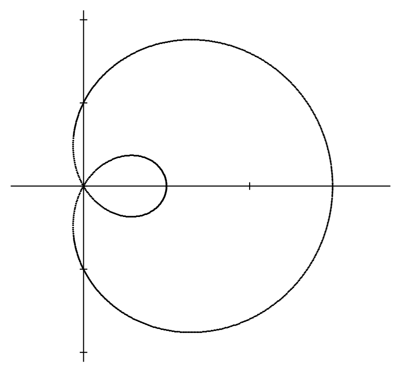
37. Evaluate the integral

$$\frac{1}{2}\int_0^{2\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt.$$

along the curve parametrized by

$$x = (1 + 2\cos t)\cos t$$
$$y = (1 + 2\cos t)\sin t$$

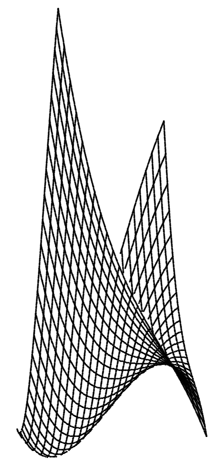
in the diagram. Interpret your result.



38. The surface $z = xy$ is a saddle. Through any point on this surface, there pass two straight lines that lie on the surface. Confirm that this is true. In particular, find equations for the two lines that pass through the point $(2, 1, 2)$.

39. You are of course familiar with the formulas $x = r\cos\theta$ and $y = r\sin\theta$ (which convert polar coordinates into Cartesian coordinates), and also with the formulas $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ (which convert Cartesian coordinates into polar coordinates). Use these to calculate $\dfrac{\partial x}{\partial \theta}$ (which is a function of r and θ) and $\dfrac{\partial \theta}{\partial x}$ (which is a function of x and y). Are the results what you expected?

40. Consider the function defined by $f(x, y) = x^3 + y^3 - 3xy$.
(a) Find all the critical points of f.
(b) Decide whether any of them correspond to extreme values of the function.
(c) Show that the surface $z = f(x, y)$ has a straight line ruled on it.



41. The figure shows the part of the surface $z = x^3 + y^3 - 3xy$ defined for $-0.6 \le x \le 1.4$ and $-0.6 \le y \le 1.4$. Use this to help you sketch a system of level curves $z = k$ for the surface. By the way, the case $k = -1$ is interesting.

42. Find the point on the plane $ax + by + cz = d$ that is closest to the origin. What is the distance from the origin to this plane? Your answers will of course depend on a, b, c, and d.

43. The surface $z^2 = 4x^2 + 4y^2$ is a familiar geometric object. What is it called?

44. The point $P = (1.6, 1.2, 0.936)$ lies on the space curve that is parametrized by $(x, y, z) = \mathcal{C}(t) = (2\cos t, 2\sin t, \sin 3t)$. Explore the meaning of the terminology *center of curvature* of this curve at P.

45. Consider the function F defined by $F(x,y) = \dfrac{2y}{x^2 + y^2 + 1}$ for all points in the xy-plane.
(a) Show that all the level curves for the surface $z = F(x, y)$ are circles. Make a diagram.
(b) Show that F has two critical points — one is a local maximum, and the other is a local minimum.
(c) Show that the unit circle provides the route of steepest ascent from $(0.8, -0.6, -0.6)$ to $(0.28, 0.96, 0.96)$.

46. The point $P = (2, 1, 3)$ is on the *hyperboloid* $x^2 + 4y^2 + 1 = z^2$.
(a) Find an equation for the plane that is tangent at P to this surface.
(b) Explain the descriptive name for the surface.
(c) Show that there is *no* line through P that lies completely on this surface.

47. Verify that the point $Q = (7, 2, 8)$ is on the hyperboloid $x^2 + 4y^2 - 1 = z^2$.
(a) Show that every level curve $z = k$ is an ellipse.
(b) Conclude that this hyperboloid is a connected surface, in contrast to the preceding example, which had two separate parts. (Using the classical terminology, this is a *one-sheeted* hyperboloid, and the preceding example is a *two-sheeted* hyperboloid.) Make a sketch of the surface that is consistent with your findings.
(c) Show that the line $(x, y, z) = (7 + 8t, 2 + 3t, 8 + 10t)$ lies completely on the hyperboloid.
(d) Find another line through Q that also lies completely on this surface.

48. Given functions $P(x,y)$ and $Q(x,y)$, and a path $\mathcal{C} : (x,y) = (x(t), y(t))$ parametrized for $a \le t \le b$, the integral formula $\displaystyle\int_a^b \left(P(x,y)\frac{dx}{dt} + Q(x,y)\frac{dy}{dt} \right) dt$, usually abbreviated to just $\int_{\mathcal{C}} P\,dx + Q\,dy$, is called a *line integral.* How is the value of a line integral affected if the path $\mathcal{C}$ is replaced by tracing its curve in the opposite direction? Explain.

49. Suppose that $[P(x,y), Q(x,y)]$ is a gradient field, and that $\mathcal{C}$ is a piecewise differentiable path in the xy-plane. It so happens that the value of $\int_{\mathcal{C}} P\,dx + Q\,dy$ depends only on the endpoints of the curve traced by $\mathcal{C}$.
(a) Verify this for the field $[xy^2, x^2y]$ by selecting at least two different piecewise differentiable paths from $(0, -1)$ to $(1, 1)$ and evaluating both integrals.
(b) Use the Chain Rule to prove the assertion in generality.

49.
50. What is the value of a line integral $\int_{\mathcal{C}} P\,dx + Q\,dy$ when $[P(x,y), Q(x,y)]$ is a gradient field, and the integration path $\mathcal{C}$ is *closed*?

51. Consider the functions $P(x,y) = \dfrac{-y}{x^2+y^2}$ and $Q(x,y) = \dfrac{x}{x^2+y^2}$, which are defined at all points of the xy-plane except the origin. Is $[P, Q]$ a gradient field in this region?

52. Evaluate $\int_{\mathcal{C}} P\,dx + Q\,dy$, where $[P, Q]$ is the field in the preceding question, and $\mathcal{C}$ is the unit circle traced in a counterclockwise sense. Hmm . . .

53. A single curve can be parametrized in infinitely many ways. For example, consider the upper unit semicircle $(x,y) = (\cos t, \sin t)$ for $0 \le t \le \pi$. This arc can also be parametrized rationally by the equation $(x,y) = \left(\dfrac{-2t}{1+t^2}, \dfrac{1-t^2}{1+t^2} \right)$ for $-1 \le t \le 1$. Verify that this is true, and show that one parametrization proceeds with constant speed, while the other does not. If a line integral calculation involved this semicircular arc, would it matter which of these parametrizations was used? Explain.

54. (Continuation) Invent another parametrization of the upper unit semicircle.

55. Consider a fluid flowing over a region $\mathcal{D}$ of the plane. This can be modeled by using a vector function $\vec{v}(x,y) = [P(x,y), Q(x,y)]$ to indicate a velocity vector for the fluid particle located at point (x,y) in $\mathcal{D}$. We call such a representation a *vector field* on $\mathcal{D}$. Sketch the following vector fields for at least four points in the region $\{(x,y) : x^2 + y^2 \le 16\}$ and see if you can understand the overall nature of the fluid flow:
(a) $\vec{v}(x,y) = [x, y+1]$ **(b)** $\vec{v}(x,y) = [-y, x]$

56. The *vector field* $\left[\dfrac{-x}{(x^2+y^2)^{3/2}}, \dfrac{-y}{(x^2+y^2)^{3/2}} \right]$ is a gradient field. Show this by finding an "antiderivative" for it. Such an antiderivative is often called a *potential function*.

57. (Continuation) The preceding vector field is an example of an *inverse-square* field. Explain the terminology.

58. The equation $8(x-2)+9(y+1)+12(z-7)=0$ represents a plane. The left side of the equation also has the look of a dot product. Make use of this observation to explain why the vector $[8, 9, 12]$ is perpendicular to this plane.

59. The line $(x, y, z) = (5+3t, 1+2t, 13+6t)$ intersects the plane that was given in the previous question. What angle does the line make with the plane?

60. The length of vector $\mathbf{u}$ is m, and the length of vector $\mathbf{v}$ is n, and the angle formed by $\mathbf{u}$ and $\mathbf{v}$ is θ. In terms of m, n, and θ, write the value of $\mathbf{u}\bullet\mathbf{v}$. For what θ will the value of the dot product be as large as it can be?

61. Find components for a nonzero vector that is perpendicular to both $[3, 2, 6]$ and $[8, 9, 12]$.

62. Find components for the vector that is obtained by (perpendicularly) projecting $[3, 2, 6]$ onto the direction defined by $[8, 9, 12]$.

63. Find components for the vector that is obtained by (perpendicularly) projecting $[3, 2, 6]$ onto the plane $8(x-2)+9(y+1)+12(z-7)=0$.

64. A particle moves through three-dimensional space according to the parametric equation $(x, y, z) = (2\cos t, 2\sin t, \sin 3t)$. At what position(s) is the particle moving fastest? At what position(s) is the particle moving slowest?

65. *Vectors in the abstract.* Suppose that $f(t)$ and $g(t)$ are each defined continuously for $0 \le t \le 1$. One of the interesting ways to look at a function is to regard it as an infinite-dimensional vector, where each of its values is a separate component! With this startling idea in mind, how would you interpret $\int_0^1 f(t)g(t)\,dt$?

66. (Continuation) Find an interpretation for $\sqrt{\displaystyle\int_0^1 f(t)^2\,dt}$.

67. Evaluate the line integral $\int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz$, given that $[\,P, Q, R\,]$ is the vector field $\left[\dfrac{x}{(x^2+y^2+z^2)^{3/2}}, \dfrac{y}{(x^2+y^2+z^2)^{3/2}}, \dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right]$, and $\mathcal{C}$ is the closed path defined by $(x, y, z) = (\sin t - \cos t, \sin t + \cos t, \cos 2t)$ for $0 \le t \le 2\pi$.

68. For each positive integer n, and for any $0 \le t \le 1$, let $s_n(t) = \sqrt{2}\sin(\pi n t)$. Using the sum to product rule below, evaluate $\int_0^1 s_n(t)s_m(t)\,dt$. The special case $n = m$ is interesting.

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$$

69. Given a point P on a curve parametrized by $(x, y, z) = (f(t), g(t), h(t))$ in $\mathbf{R}^3$, the *normal plane* at P is the plane that contains P and that is perpendicular to the curve, which means perpendicular to the tangent vector $[f'(t), g'(t), h'(t)]$.

Consider the *helix* defined by $(x, y, z) = \mathcal{H}(t) = (\cos t, \sin t, t)$.
(a) Write an equation for the normal plane at time $t = a$.
(b) Write an equation for the normal plane at a slightly later time $t = b$.
(c) Find coordinates for two points that lie on the line where these two planes intersect.
(d) Create a vector that points in the direction of this line.
(e) Consider what happens to your vector as b approaches a.

70. *New notation for vectors.* In many calculus books, the unit vectors in the coordinate-axis directions are given special names:

$$\begin{aligned}\mathbf{i} &= [1, 0, 0]\\ \mathbf{j} &= [0, 1, 0]\\ \mathbf{k} &= [0, 0, 1]\end{aligned}$$

which means that $[a, b, c]$ can be written $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Using this notation to express your answer, find three mutually perpendicular vectors — other than $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ — that all have the same length.

71. *The cross product.* Given two vectors $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, there are infinitely many vectors $[a, b, c]$ that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It is a routine exercise in algebra to find one, and it requires that you make a choice during the process. It so happens that there is a "natural" way to make this choice, and an interesting formula results.
(a) Confirm that $\mathbf{w} = [\,qf - re, rd - pf, pe - qd\,]$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
(b) It is customary to call $\mathbf{w}$ the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, and to write $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Is it true that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$?
(c) Show that the length of $\mathbf{u} \times \mathbf{v}$ is equal to the length of $\mathbf{u}$ times the length of $\mathbf{v}$ times the the sine of the angle formed by $\mathbf{u}$ and $\mathbf{v}$.
(d) Give three explanations of the fact $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Also explain why the zero is in boldface type.

72. Find the length of the helical arc $(x, y, z) = (a \cos t, a \sin t, bt)$ for $0 \le t \le 2\pi$.

73. Find the area of the triangle whose vertices are $A = (1, 2, 3)$, $B = (7, 5, 5)$, and $C = (8, 6, 7)$.

74. The curve parametrized by $(x, y, z) = (t \cos 2\pi t, t \sin 2\pi t, 2t)$ lies on the conical surface $z^2 = 4x^2 + 4y^2$. Confirm this, and make a sketch. Show that the curve goes through the point $(0, 2.25, 4.5)$. Show that its velocity vector at that point is tangent to the surface.

75. *More on the cross product.* You now have a complicated formula that finds a vector that is perpendicular to both $\mathbf{i}$ and $\mathbf{j}$. Confirm the following special cases:

$$\begin{aligned} \mathbf{i} &= \mathbf{j} \times \mathbf{k} \\ \mathbf{j} &= \mathbf{k} \times \mathbf{i} \\ \mathbf{k} &= \mathbf{i} \times \mathbf{j} \\ -\mathbf{i} &= \mathbf{k} \times \mathbf{j} \\ -\mathbf{j} &= \mathbf{i} \times \mathbf{k} \\ -\mathbf{k} &= \mathbf{j} \times \mathbf{i} \end{aligned}$$

Use these results and the usual rules of algebra to derive the rest of the formula for

$$(p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}).$$

76. In multivariable calculus, one meets many types of functions. For example, we have encountered real-valued functions of position (the temperature distribution on a hotplate), vector-valued functions of position (the gradient of a real-valued function of position), and point-valued functions of time (paths in the xy-plane or in xyz-space). One can also view paths as vector-valued functions of t, by filling in the vectors that reach from the origin to the points on the curve. In other words, $(x(t), y(t), z(t))$ and $x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ describe essentially the same thing.

One way to classify the different functions is to focus on the dimensions of the domain and range spaces. For example, a curve in xyz-space is a correspondence from 1 dimension to 3 dimensions, symbolized $\mathbf{R}^1 \to \mathbf{R}^3$. Classify some other examples of functions.

77. In multivariable calculus, one encounters many versions of the product rule for differentiation. For example, suppose that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are vector-valued functions of t. Then $f(t) = \mathbf{u}(t) \bullet \mathbf{v}(t)$ is a real-valued function of t. Propose a *product rule* that covers this situation, and test it on the vector functions $\mathbf{u}(t) = t^2\mathbf{i} + (2 - \cos t)\mathbf{j}$ and $\mathbf{v}(t) = (\sin t)\mathbf{i} + t\mathbf{j}$.

78. In xyz-space, there is of course a rule for differentiating the cross product of vectors. Formulate the rule and test it on an example of your invention.

79. Let $\mathbf{u}(t)$ be a vector that depends on t, and let f be a real-valued function of t. Then $\mathbf{v}(t) = f(t)\mathbf{u}(t)$ defines another vector-valued function of t. Write out a product rule for $\mathbf{v}'(t)$, and test it on the example $\mathbf{u}(t) = [\cos t, \sin t]$ and $f(t) = 2^t$.

80. Suppose that $\mathbf{u}(t)$ depends on t (thus $\mathbf{u}$ is a path that traces a curve) in such a way that $\mathbf{u}(t) \bullet \mathbf{u}(t)$ has a constant value. Show that $\mathbf{u}(t)$ is perpendicular to $\mathbf{u}'(t)$. Give a familiar geometric interpretation of this result.

81. A path $\mathbf{u}$ is called "smooth" provided that it is differentiable (meaning that $\mathbf{u}'(t)$ is defined throughout the path), and $\mathbf{u}'(t)$ is never $\mathbf{0}$. For example, show that $\mathbf{u}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is not smooth. What is the significance of the non-smoothness?

82. Any path $\mathbf{p}$ whose trace (image) does not go through the origin can be expressed in the form $r(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ is a unit vector for all t. Explain why, and apply this concept to the example $\mathbf{p}(t) = [t, 1]$.

83. Calculate $(a\mathbf{i} + b\mathbf{j}) \bullet (a\mathbf{i} + b\mathbf{j})$. Is the result familiar?

84. Calculate $(a\mathbf{i} + b\mathbf{j}) \times (c\mathbf{i} + d\mathbf{j})$. Could you have predicted the result?

85. Draw a diagram of a differentiable path $\mathbf{p}$. Remember that each $\mathbf{p}(t)$ is a *position vector* that connects the origin to a point on the curve. Use your diagram to illustrate the three vectors

$$\mathbf{p}(t+h) - \mathbf{p}(t),$$
$$\frac{1}{h}\left(\mathbf{p}(t+h) - \mathbf{p}(t)\right), \text{ and}$$
$$\lim_{h\to 0} \frac{\mathbf{p}(t+h) - \mathbf{p}(t)}{h}.$$

86. Suppose that

$\mathbf{u}(t)$ is a unit vector for all t,
the length of $\mathbf{u}'(t)$ is a positive constant n, and
$r(t)$ is differentiable so that $r'(t) = mr(t)$ for some constant m and all t.

Then the path defined by $\mathbf{p}(t) = r(t)\mathbf{u}(t)$ has the property that $\mathbf{p}(t)$ and $\mathbf{p}'(t)$ make the same angle for all t. Show that this is true, by expressing the cosine of this angle in terms of the constants m and n. Have you encountered examples like this before?

87. Given a path $\mathbf{p}$, let $\mathbf{T}(t) = \dfrac{\mathbf{p}'(t)}{|\mathbf{p}'(t)|}$ be its *unit tangent vector*. Explain why $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. Confirm this for the helix defined by $\mathbf{p}(t) = (a\cos t, a\sin t, bt)$.

88. Interpret the following instance of the *Chain Rule*

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

in terms of the concepts you have met this term. In particular, use as much technical vocabulary (gradient, directional derivative, dot product, level surface, velocity, etc) as you can. In your discussion, consider the two special instances

$$\frac{dF}{dt} = 0 \quad \text{and} \quad 1 = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

89. Given that $\mathbf{u}'(t)$ is a constant vector $\mathbf{m}$, what do you deduce about the path $\mathbf{u}$?

90. The derivative of a function at a point is a *local multiplier*, which relates small changes in domain values to correspondingly small changes in range values. In traditional symbols,

$$\begin{aligned} f(p+\Delta p) - f(p) &\approx f'(p)\Delta p \\ \Delta f(p) &\approx f'(p)\Delta p. \end{aligned}$$

It is important to notice that the new examples we have met do not always allow us to divide both sides of the approximation by Δp, which has been a common practice up until now. Explain.

91. Consider a function T of the type $\mathbf{R}^2 \to \mathbf{R}^1$ (temperature on a hotplate, for example). Suppose that the temperature $T(8,5) = 137$, and the partial derivatives $T_x(8,5) = 6$ and $T_y(8,5) = -4$ are known, and suppose that an estimate of $T(8.3, 5.7)$ is required. Calculate one.

92. (Continuation) Explain why the gradient vector $\nabla T(8,5) = [6, -4]$ serves as the local multiplier that was needed to convert $\Delta p = [0.3, 0.7]$ into $\Delta T(8,5) = -1.0$. Thus it makes sense to regard ∇T as "the derivative" of T in this instance, and the terminology "partial derivative" makes more sense, too.

93. It will be necessary to consider "second derivatives". These have to be discussed "partially" when considering functions of the type $\mathbf{R}^2 \to \mathbf{R}^1$. In other words, you will see that expressions such as

$$\frac{\partial}{\partial x}\frac{\partial T}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial x}\frac{\partial T}{\partial y},$$

which are also written

$$\frac{\partial^2 T}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 T}{\partial x \partial y},$$

are useful. Other notations are T_{xx} and T_{yx}.

Starting with $T(x,y) = x^y$, calculate all four *second partials* T_{xx}, T_{xy}, T_{yx}, and T_{yy}. You will of course have to calculate T_x and T_y first.

94. The temperature at point (x,y,z) is $T(x,y,z) = \dfrac{100}{x^2+y^2+z^2+1}$. A bug is flying through three-dimensional space, according to the equation $(x,y,z) = (t, t^2, t^3)$. Distances are measured in centimeters, and time is measured in seconds. Describe the rate of temperature change experienced by the bug as it passes the point $(1,1,1)$. Give two answers, one in degrees per second, and the other in degrees per centimeter.

95. Find the size of the acute angle between the radial vector $[x, y]$ and the tangent vector for the spiral traced by $[x, y] = 2^t[\cos 2\pi t, \sin 2\pi t]$. Your answer should not depend on t.

96. It is a curiosity that $2^4 = 4^2$, and a challenging exercise to produce other pairs of positive (and unequal) numbers that fit the equation $x^y = y^x$. You can use the equation-solving capability of your calculator to find a few. You can also consider a parametric approach: What is the intersection of the line $y = mx$ and the graph of $x^y = y^x$?

97. (Continuation) Analyze the curve parametrized by $x = t^{1/(t-1)}$ and $y = t^{t/(t-1)}$. In particular, evaluate the limiting behavior of x and y as t approaches 0, 1, and ∞.

98. The surface $x^2 + y^2 - z^2 = 0$ is a cone, and the surface $x^2 + y^2 - z^2 = -1$ consists of two separated pieces. The surface $x^2 + y^2 - z^2 = 1$ consists of one connected piece. Discuss these and other level surfaces for the function $T(x, y, z) = x^2 + y^2 - z^2$, and make a sketch.

99. (Continuation) Describe a curve that goes through the point $(1, 1, 1)$ and that intersects all of the level surfaces perpendicularly. Find two different parametrizations (paths) that trace this curve.

100. *Using perpendicularity.* Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, and that $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$, and $|\mathbf{w}| = 1$. Calculate $(a\mathbf{u} + b\mathbf{v} + c\mathbf{w})\bullet(a\mathbf{u} + b\mathbf{v} + c\mathbf{w})$, in terms of the scalars a, b, and c.

101. Let $\mathbf{p}(t) = \dfrac{-2t}{1+t^2}\,\mathbf{i} + \dfrac{1-t^2}{1+t^2}\,\mathbf{j}$. Verify that $|\mathbf{p}(t)| = 1$ for all t. Find $\mathbf{p}(t)\bullet\mathbf{p}'(t)$.

102. Given a path $\mathbf{p}$, it is customary to borrow terminology from physics and call $\mathbf{p}'$ the *velocity* of $\mathbf{p}$, and to call $\mathbf{p}''$ the *acceleration* of $\mathbf{p}$. Furthermore, if it is known that the velocity is never $\mathbf{0}$, then the velocity can be expressed $\mathbf{p}'(t) = r(t)\mathbf{T}(t)$, where $r(t) = |\mathbf{p}'(t)|$. The vector $\mathbf{T}(t)$ is usually called the *unit tangent* for $\mathbf{p}(t)$. Why? What is $r(t)$ usually called? Find $\mathbf{T}(t)$ and $r(t)$ for the example in the previous question.

103. (Continuation) Apply the product rule to $\mathbf{p}'(t) = r(t)\mathbf{T}(t)$, to calculate the acceleration in terms of $r(t)$, $\mathbf{T}(t)$, and their derivatives. This expresses the acceleration vector in terms of two *orthogonal* (that means perpendicular) contributions — one resulting from a change of speed, and the other resulting from a change of direction. Explain.

104. (Continuation) It is intuitive that acceleration due to changing direction is dependent on two things — the speed and the curvature of the path. To better appreciate this, consider the simple example $\mathbf{p}(t) = [r\cos kt, r\sin kt]$, where r and k are positive constants. In this example, speed is constant — it is only direction that varies. Calculate $\mathbf{p}'(t)$ and $\mathbf{p}''(t)$, and notice that they are orthogonal. Also verify that $|\mathbf{T}'| = v/r$ and $|\mathbf{p}''| = v^2/r$, where v is the speed.

105. Consider the parabolic curve $y = x^2$, which goes through the point $P = (1, 1)$.
(a) In terms of c, write an expression for the line that is orthogonal to the curve at (c, c^2).
(b) Assuming that $c \neq 1$, find the intersection R_c of this line with the line $y = \frac{1}{2}(3 - x)$, which is orthogonal to the curve at P.
(c) The intersection point R_c depends on the value of c. Find the limiting position of R_c as c approaches 1. This is called the *center of curvature* of the parabola at P.
(d) Explain the terminology, and calculate the *radius of curvature* of the parabola at P.

Apply questions **2** through **7** to the curve parametrized by $\mathbf{p}(t) = [t, t^2]$:

106. Calculate the velocity vector $\mathbf{v} = \mathbf{p}'$, the acceleration vector $\mathbf{a} = \mathbf{v}' = \mathbf{p}''$, and the unit tangent vector $\mathbf{T} = \dfrac{1}{|\mathbf{v}|}\mathbf{v}$.

107. The function $|\mathbf{v}|$ is often written as $\dfrac{ds}{dt}$, and is called the *speed* of the parametrization. Its derivative $\dfrac{d^2s}{dt^2}$ is called the *scalar acceleration*. Calculate it.

108. Explain why $\mathbf{a} = \dfrac{d^2s}{dt^2}\,\mathbf{T} + \dfrac{ds}{dt}\,\mathbf{T}'$ is expected, then calculate $\mathbf{T}'$ as confirmation.

109. You should expect that $\mathbf{T}'(1)$ points from P toward the center of curvature of the parabolic arc traced by $\mathbf{p}(t)$. Explain why.

110. In general, $\mathbf{T}'$ is not a unit vector, but it can be expressed in the form $m\mathbf{N}$, where $\mathbf{N}$ is the *unit normal vector* and m is the magnitude of $\mathbf{T}'$. Calculate m and $\mathbf{N}$ for the current example.

111. Verify that $|\mathbf{T}'| = \dfrac{v}{r}$ at $P = (1, 1)$, where v is the speed $\dfrac{ds}{dt}$ and r is the radius of curvature calculated in question 1.

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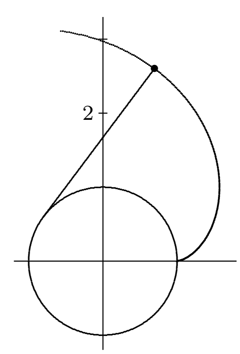
112. In a limiting sense, curvature describes the rate at which direction changes with respect to distance traveled. According to this definition, the curvature of a circle of radius r is $1/r$ (at each of its points). Explain.

113. The diagram shows a snapshot of a thread that is being unwound from a spool of radius 1, represented by the unit circle $x^2 + y^2 = 1$. As the diagram suggests, the end of the thread was initially at $(1, 0)$. Let $(\cos t, \sin t)$ be the point where the thread is tangent to the spool. Write an equation for the position of the end of the thread in terms of t. The spiral traced by the end of the thread is called the *involute* of the unit circle.



114. Show that $\mathbf{w} = [3, 2]$ can be expressed as a *linear combination* $a\mathbf{u}+b\mathbf{v}$ of the *orthonormal* vectors $\mathbf{u} = [0.8, 0.6]$ and $\mathbf{v} = [-0.6, 0.8]$ in one and only one way. In other words, find the unique values a and b.

115. Verify that the vectors $\mathbf{u}_1 = \left[\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right]$ $\mathbf{u}_2 = \left[\frac{3}{7}, -\frac{6}{7}, \frac{2}{7}\right]$ $\mathbf{u}_3 = \left[\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right]$ are orthonormal. Verify also that $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$, which shows that the three vectors form a *right-handed* basis for coordinates in $\mathbf{R}^3$. Finally, show that $\mathbf{w} = [5, 1, 0]$ can be expressed uniquely in the form $a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$.

116. Given a curve parametrized by $[x, y] = \mathbf{p}(t)$, its *radius of curvature* is defined to be

$$r(t) = \frac{|\mathbf{v}(t)|}{|\mathbf{T}'(t)|}\,,$$

where $\mathbf{v} = \mathbf{p}'$ and $\mathbf{T}$ is the unit tangent $\dfrac{\mathbf{v}}{|\mathbf{v}|}$. Explain the logic behind this definition.

117. Recall the definition $\mathbf{N} = \dfrac{\mathbf{T}'}{|\mathbf{T}'|}$. Explain why $\mathbf{N}$ is perpendicular to $\mathbf{T}$.

118. Recall that the acceleration vector is defined to be $\mathbf{a} = \mathbf{v}'$. Explain why

$$\mathbf{a} = \frac{d^2s}{dt^2}\,\mathbf{T} + \frac{1}{r}\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

119. The convenient formula $r = \dfrac{|\mathbf{v}|^3}{|\mathbf{v} \times \mathbf{a}|}$ for the radius of curvature is a simple consequence of the orthogonality of vectors in the preceding. Verify the formula, then use it to find the radius of curvature of
(a) the parabolic curve traced by $\mathbf{p}(t) = [t, t^2]$;
(b) the helical curve traced by $\mathbf{p}(t) = [b\cos t, b\sin t, mt]$, where m and b are positive constants.

120. Find the radii of curvature of the ellipse $(x, y) = (a\cos t, b\sin t)$ at the points where it crosses the x- and y-axes.

121. Given a curve in $\mathbf{R}^3$ that has been parametrized by a path $\mathbf{p}$, let $\mathbf{T}$ be the unit tangent vector and $\mathbf{N}$ be the unit normal vector. It is customary to call $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ the *torsion vector*.
(a) Explain why $\mathbf{B}'$ is perpendicular to $\mathbf{B}$.
(b) Apply the product rule to $\mathbf{B}\bullet\mathbf{T}$ to show that $\mathbf{B}'$ is also perpendicular to $\mathbf{T}$.

122. Show that the radius of curvature of $y = f(x)$ is $\dfrac{\left(1 + (f'(x))^2\right)^{3/2}}{|f''(x)|}$.

123. Suppose that $\mathbf{u}$ is a unit vector. Explain why the length of the projection of any vector $\mathbf{v}$ onto $\mathbf{u}$ is exactly $\mathbf{v} \bullet \mathbf{u}$.

124. It is not surprising that the radius of curvature of $\mathbf{p}(t)$ is undefined (infinite) if $\mathbf{p}(t)$ is a linear function. Show that this phenomenon also occurs for nonlinear examples such as $\mathbf{p}(t) = [t, t^3]$ and $\mathbf{q}(t) = [t^3, t^6]$. Why might the second example be surprising?

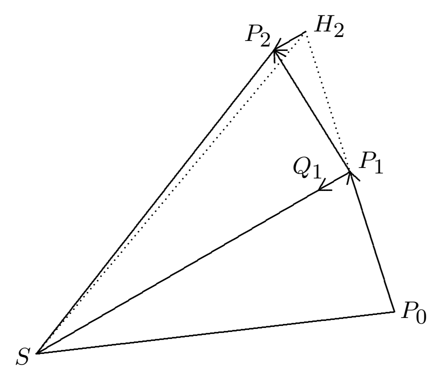
125. Suppose that the torsion vector $\mathbf{B}$ for a path $\mathbf{p}$ is constant. What does this say about the path traced by $\mathbf{p}$? Give an example to illustrate your conclusion.

126. Find the radius of curvature of the cycloid $\mathbf{p}(t) = [t - \sin t, 1 - \cos t]$. Where on the curve is the largest radius found? Where on the curve is the smallest radius found?

127. Find the radius of curvature of the curve $y = \sqrt{x}$. Compare this curve with the cycloid in the vicinity of the origin.

128. Because there are many ways of parametrizing a given curve, it is conceivable that the various formulas for the radius of curvature could assign different values to the same point! Do an example that illustrates that this need not happen, then try to explain why it cannot ever happen.

129. The diagram shows a polygonal approximation to the orbit of a planet moving around the Sun S. Points P_0 and P_1 are two positions of the planet that are separated by Δt units of time. Point H_2 is the position the planet would reach during the next time interval Δt if there were no gravitational force acting, and P_2 is the actual position reached, because the gravitational force from S causes a velocity change $\Delta\mathbf{v} = P_1Q_1/\Delta t$. Thus vectors P_0P_1 and P_1H_2 are the same, vectors P_1Q_1 and H_2P_2 are the same, and P_1, Q_1, and S are collinear. Explain why the areas of triangles SP_0P_1, SP_1H_2, and SP_1P_2 are the same. By letting Δt approach 0, deduce *Kepler's Second Law*: The radius vector from the Sun to one of its planets sweeps out equal areas in equal times.



130. Let $\mathbf{r}(t)$ be the radius vector from the Sun to one of its planets, and let $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{v}'$. Assume that gravity is a *central force*, which means simply that $\mathbf{a}$ is always some negative multiple of $\mathbf{r}$.
(a) Apply the Product Rule to show that $\mathbf{r} \times \mathbf{v}$ is a constant vector.
(b) Explain why $\frac{1}{2}|\mathbf{r} \times \mathbf{v}|$ is the rate at which the Sun-planet radius vector sweeps out area. Show that this implies Kepler's Second Law.

131. Given the acceleration vectors $\mathbf{p}''(t) = [6t, \cos t]$, the velocity vector $\mathbf{p}'(0) = [1, 2]$, and the position vector $\mathbf{p}(0) = [-\pi^3, -1]$, calculate the position vector $\mathbf{p}(\pi)$.

132. There is yet another curvature formula that one often sees in the AP Calculus curriculum: Given a parametrized curve $(x(t), y(t))$, its radius of curvature is

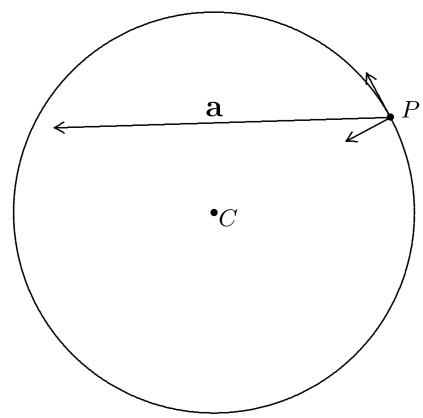
$$r(t) = \frac{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}{|x'(t)y''(t) - y'(t)x''(t)|}\,.$$

Prove this formula, using what you have learned this term.

133. Find the area of the triangle PEA that is determined by the points $P = (0, 3, 8)$, $E = (7, 7, 13)$, and $A = (8, -1, 18)$.

134. Given the cycloid $\mathbf{p}(t) = [t - \sin t, 1 - \cos t]$, find explicit, simplified formulas for its unit tangent vector $\mathbf{T}(t)$ and its unit normal vector $\mathbf{N}(t)$. In particular, evaluate $\mathbf{T}(\pi/2)$ and $\mathbf{N}(\pi/2)$.

135. The diagram shows the circle traced parametrically by a path $\mathbf{p}$, as well as the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{a}$ at one of the points P on the circle, whose center is C. Obtaining all numerical data by *measuring the diagram*, calculate plausible values for the radius of curvature, and for the derivatives $\dfrac{ds}{dt}$ and $\dfrac{d}{dt}\left(\dfrac{ds}{dt}\right)$ at P.



136. The *evolute* of a curve consists of its centers of curvature. Show that the evolute of a cycloid is yet another cycloid!

137. An object that moves in response to a central force must have a *planar* orbit. In other words, if it is given that $\mathbf{p}''(t)$ and $\mathbf{p}(t)$ are always parallel, then the curve traced by $\mathbf{p}(t)$ must lie in a plane. Prove this statement.

138. The depth (in feet) of Lake Mathematica is $f(x, y) = 300 - 2x^2 - 3y^2$. A swimmer at $(2, 1)$ is swimming in the direction that decreases the depth most rapidly.
(a) What direction is this?
(b) The swimmer is moving through the water at 4 fps. At what rate (in fps) is the depth of the water beneath the swimmer decreasing?

139. Given that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, where $|\mathbf{u}| = 1 = |\mathbf{v}|$ and $\mathbf{u} \bullet \mathbf{v} = 0$, express the number $|\mathbf{w}|$ in terms of the numbers a and b.

140. Given three arbitrary vectors **u**, **v**, and **w** in $\mathbf{R}^3$, does the associative law

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

hold? In other words, can the parentheses be omitted?

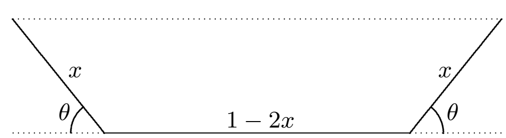
141. Given three arbitrary vectors **u**, **v**, and **w** in $\mathbf{R}^3$, their *triple scalar product* is defined to be $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$.
(a) Explain the use of the word "scalar" in the definition.
(b) Show how to interpret the value of this expression as a *volume*.
(c) Show that the same value is obtained from $\mathbf{v} \bullet (\mathbf{w} \times \mathbf{u})$ or $\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$.

142. The generic equation for uniform circular motion is $\mathbf{p}(t) = r[\cos kt, \sin kt]$. Given a particular radius r, find (in terms of r) the value of k that makes the length of $\mathbf{a} = \mathbf{p}''$ equal to r^{-2}. This formula for angular speed k solves a special version of the inverse-square law, which requires only that the length of $\mathbf{p}''$ be proportional to r^{-2}.

143. (Continuation) This formula for $k(r)$ applies to a hypothetical planetary system. Show that Kepler's Third Law is satisfied by circular orbits in this system: In other words, the square of a planet's period is proportional to the cube of the radius of its orbit.

144. Of all the points on the plane $3x + 4y + z = 52$, find the one that is closest to the origin. Instead of the usual approach, however, let $f(x, y)$ be the square of the distance from $(x, y, 52 - 3x - 4y)$ to the origin. Find the desired point by calculating the partial derivatives f_x and f_y, then using these to find the only critical point of f.

145. The figure shows an end view of a long strip of metal that has been bent to form a channel. The objective is to maximize the carrying capacity of this channel, which means that the cross-sectional area that is shown should be as large as possible. The width of the strip is 1 meter, as shown in the diagram. There are two variables in the figure; one is the width x of the two equal outer sections, and the other is the bending angle θ. Use these two variables to express the area of the trapezoid as a function $A(x, \theta)$. Then calculate the partial derivatives A_x and A_θ and look for critical points. Notice that the meaningful values for x and θ lie in the domain rectangle $0 \le x \le \frac{1}{2}$ and $0 \le \theta \le \frac{\pi}{2}$. Also be alert for algebraic shortcuts when solving the equations $A_x = 0$ and $A_\theta = 0$.



146. Given that **u** and **v** are perpendicular vectors of unit length in $\mathbf{R}^3$, and that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, show that $\mathbf{u} = \mathbf{v} \times \mathbf{w}$, and then that $\mathbf{v} = \mathbf{w} \times \mathbf{u}$.

147. It is convenient to use polar coordinates to describe conic sections, by placing one of the focal points at the origin (the pole). For example, the equation

$$r = \frac{b^2}{a + c\cos\theta}$$

describes an ellipse with symmetry axis lengths $2a$ and $2b$, and focal separation $2c$, provided that $a^2 = b^2 + c^2$. Verify that this is so.

148. Apply the Product Rule to $\mathbf{r}(t) = r(t)\mathbf{u}(t)$, where $\mathbf{u}$ is a unit vector. This expresses velocity in terms of a "radial" component and an "angular" component. Explain.

149. In $\mathbf{R}^2$, unit vectors $\mathbf{u}$ can be considered functions of θ. Justify the equation $\mathbf{u}' = \dfrac{d\theta}{dt}\mathbf{u}_\theta$, and comment on the notation $\mathbf{u}_\theta$.

150. Use the orthonormality of $\mathbf{u}$ and $\mathbf{u}_\theta$ to show that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\,\frac{d\theta}{dt}\right)^2},$$

which describes speed in terms of polar coordinates.

151. Kepler's Laws are applicable to central forces that obey the *inverse-square law*, which means that $\mathbf{a} = \mathbf{r}'' = -\dfrac{g}{r^2}\mathbf{u}$, for some positive constant g. Explain the notation.

152. If the force that produces $\mathbf{a}$ is central, then we have shown that $\mathbf{r} \times \mathbf{r}'$ is a constant vector. Without loss of generality, let us assume that the constant vector is $h\mathbf{k}$, for some positive constant h, so that the planetary orbit lies in the xy-plane. Assume also that the central force is directed toward the origin. This encourages the use of polar coordinates. Show that Kepler's Second Law can be stated in the form $r^2\dfrac{d\theta}{dt} = h$. This is remarkable because it applies to *all* orbits, not just the trivial circular ones that have constant values for $\dfrac{d\theta}{dt}$ and r.

153. The orthonormal basis $\mathbf{u}$, $\mathbf{u}_\theta$, and $\mathbf{k}$ has been labeled so that $\mathbf{u} \times \mathbf{u}_\theta = \mathbf{k}$. Explain why this implies that $\mathbf{k} \times \mathbf{u} = \mathbf{u}_\theta$ and $\mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}$.

154. Justify each of the following lines:

$$\begin{aligned}
-\mathbf{u} \times \mathbf{k} &= \mathbf{u}_\theta \\
-\mathbf{u} \times \frac{d\theta}{dt}\mathbf{k} &= \mathbf{u}' \\
-g\mathbf{u} \times \frac{d\theta}{dt}\mathbf{k} &= g\mathbf{u}' \\
\mathbf{a} \times h\mathbf{k} &= g\mathbf{u}'
\end{aligned}$$

It follows that $(\mathbf{v} \times h\mathbf{k}) - g\mathbf{u}$ is a constant vector, which can be denoted $g\mathbf{e}$. Explain. Let $m = |\mathbf{e}|$. From now on, it will be assumed that $\mathbf{e}$ points in the direction of positive x-values. This enables the use of polar coordinates in the plane of the orbit.

155. *Kepler's First Law*: Justify the equation $\mathbf{r}\bullet(\mathbf{v} \times h\mathbf{k}) = \mathbf{r}\bullet(g\mathbf{u} + g\mathbf{e})$, then evaluate both sides of it to obtain $h^2 = gr + gmr\cos\theta$. Show that this is the equation of a conic section, whose major axis is aligned with $\mathbf{e}$, and whose eccentricity is m.

156. What are $\mathbf{e}$ and m in the special case of a circular orbit?

157. You now know that a planetary orbit is described by $\mathbf{r} = r\mathbf{u} = \dfrac{h^2/g}{1 + m\cos\theta}\mathbf{u}$, where h is twice the rate at which the radial vector $\mathbf{r}$ sweeps out area, g is a physical constant that is proportional to the mass of the Sun, $\mathbf{u} = [\cos\theta, \sin\theta]$, and m is the eccentricity of the orbit (which is between 0 and 1).
(a) Show that the *mean distance* from the planet to the Sun (which is half the length of the major axis of the orbit) is $\mathcal{D} = \dfrac{h^2/g}{1 - m^2}$.
(b) Show that the distance from the Sun to the center of the orbit is $m\mathcal{D}$, and half the length of the minor axis is $b = \mathcal{D}\sqrt{1 - m^2}$.
(c) Show that the length of time that it takes the planet to complete its orbit is $\mathcal{T} = \dfrac{\pi\mathcal{D}b}{h/2}$.
(d) *Kepler's Third Law* says that the square of a planet's period is proportional to the cube of its mean distance from the Sun. Confirm this by showing that $\dfrac{\mathcal{T}^2}{\mathcal{D}^3} = \dfrac{4\pi^2}{g}$.

158. Show that the length of vector $d\mathbf{v}/d\theta$ is constant — an obvious property of circular orbits that is true even for non-circular orbits. Show that the value of the constant is g/h.

159. (Continuation) Show that the vectors $\mathbf{v}$ (with their tails at the origin) trace a circle.

160. Thus far, you have seen the motion of a planet described explicitly in terms of theta. Knowing how fast a planet actually moves along a non-circular orbit requires knowing θ as a function of t. This is a complicated question.
(a) Apply the Product Rule to $\mathbf{r}' = r\mathbf{u}' + r'\mathbf{u}$. Rearrange your answer so that $\mathbf{a}$ is expressed in terms of a radial component and an angular component. (In other words, use the vectors $\mathbf{u}$ and $\mathbf{u}_\theta$ to organize your answer.) You will need to replace $\mathbf{u}'$ by $\mathbf{u}_\theta \dfrac{d\theta}{dt}$. Second derivatives of r and θ will appear in your answer.
(b) Because $\mathbf{a}$ is produced by a central force, one of the two components of $\mathbf{a}$ has to be zero. Which one? Verify that it *is* zero.

161. Consider the function $f(x, y) = 9xy(1 - x - y)$.
(a) Sketch the level curve $f(x, y) = 0$. Then sketch a few other level curves, using your intuition (no calculation needed).
(b) Find all the critical points of f. Classify each one as maximum, minimum, or saddle.

162. It is well-known that circles have a constant radius of curvature. Consider the converse statement: If a space curve has a constant radius of curvature r_0, must the curve be a (planar) circle?

163. Given an equation $z = f(x, y)$, is it possible for two level curves of f to intersect?

164. Consider the logarithmic spiral $\mathbf{p}(t) = b^t(\mathbf{i}\cos t + \mathbf{j}\sin t)$. Show that $|\mathbf{p}(t)| = b^t$ and $|\mathbf{p}'(t)| = b^t\sqrt{1 + (\ln b)^2}$.

165. Let $Q = (6, -1, 11)$, $P = (-4, 5, -5)$, $R = (7, 6, 6)$, and $S = (9, 9, 12)$. The lines PQ and RS do not intersect (they are called *skew*). Find the smallest distance that separates a point on one line from a point on the other. There are two ways to solve this question — one uses calculus, and the other does not. Find both methods.

166. To say that two skew lines in $\mathbf{R}^3$ are "perpendicular" means that their direction vectors are perpendicular. Suppose that $ABCD$ is a tetrahedron (a triangular pyramid), in which edges AB and CD are perpendicular, and edges AC and BD are perpendicular. Prove that edges AD and BC must also be perpendicular.

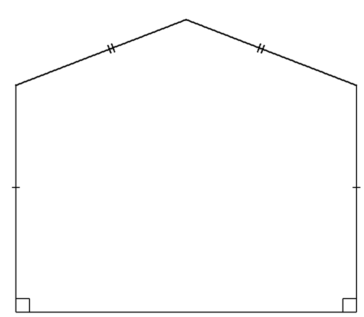
167. For each of the following, sketch level curves, and determine the nature of the critical point at the origin:
(a) $z = 2x^2 + xy - y^2$ (*Hint*: This can be factored.)
(b) $z = x^2 + 2y^2$
(c) $z = 2xy - x^2 - 2y^2$
(d) $z = x^2 - 4xy + 4y^2$ (*Hint*: This can be factored.)

168. The graph of an equation $z = ax^2 + bxy + cy^2$ is a surface that goes through the origin.
(a) Explain why the origin is a critical point for any such surface.
(b) Show that the surface intersects the xy-plane along two lines if $4ac < b^2$. Justify applying the terminology *saddle point* to the origin in this case.
(c) Show that the surface intersects the xy-plane only at the origin if $b^2 < 4ac$. What is the nature of the critical point at the origin in this case?
(d) What can be said about the remaining case, which is $b^2 = 4ac$?

169. Given a differentiable function $f(x, y)$, its gradient vector $[f_x, f_y]$ could be called the *derivative* of f, because Δf is approximated so well by $[f_x, f_y] \bullet [\Delta x, \Delta y]$. Explain.

170. Given a real-valued function $f(x, y)$, the matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ of second partial derivatives is the derivative of the gradient, thus is the *second derivative* of f. Calculate this matrix for the quadratic example $f(x, y) = ax^2 + bxy + cy^2$. Show that the determinant of the second derivative of f is $4ac - b^2$. Hmm...

171. As shown in the diagram, a 2-meter length of wire is to be bent into the shape of a pentagon that has an axis of reflective symmetry, and in which two adjacent angles are right. What is the largest area that can be enclosed by such a shape? (*Hint*: Be careful when you label the figure — some variable choices work better than others.)



172. Consider the logarithmic spiral $\mathbf{p}(t) = b^t(\mathbf{i}\cos t + \mathbf{j}\sin t)$. Show that its evolute (the locus of its centers of curvature) is itself a logarithmic spiral.

173. Let $A = (4, 0)$, $B = (0, 3)$, $C = (0, 0)$, and $P = (x, y)$. Find the coordinates of P, given that $PA + PB + PC$ is minimal. You may want to use an equation solver.

174. If we let $f(x, y)$ be the distance between (x, y) and $(3, 0)$, then $\vec{\nabla} f$ is a unit vector pointing in a predictable direction. Explain.

175. Show that if $|\mathbf{u}_1| = |\mathbf{u}_2| = |\mathbf{u}_3| = 1$ and $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$, then the angle between any two vectors $\mathbf{u}_i$ must be $120°$.

176. Show that $(0, 0)$ is a critical point of $p(x, y) = y^2 + (xy - 1)^2$. To visualize what is happening in the local vicinity of the point $(0, 0)$, it might be useful to consider the behavior of $p(x, y)$ as the point (x, y) "passes" through $(0, 0)$ along the following lines:

- the line $y = 0$ (i.e. the x-axis)
- the line $y = x$
- the line $y = 2x$
- the line $y = 3x$
- the line $y = -x$

Is $p(x, y)$ continuous at $(0, 0)$? In other words, does $\lim\limits_{(x,y)\to(0,0)} p(x, y) = p(0, 0)$?

177. Is the following function continuous at $(x, y) = (0, 0)$?

$$f(x, y) = \begin{cases} \dfrac{x - 3y}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$$

178. *The Second-Derivative Test.* Given a surface $z = f(x, y)$, and a point $P = (a, b)$ at which the gradient $[f_x(a, b), f_y(a, b)]$ is $\mathbf{0}$, one encounters the classification problem: Is $f(a, b)$ a maximal z-value, a minimal z-value, or a saddle z-value? If the second-order partial derivatives exist and are continuous, then the determinant

$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx}$$

often answers the question, because:
(a) if $0 < H(a, b)$ and $0 < f_{xx}(a, b)$, then $f(a, b)$ is a relative minimum;
(b) if $0 < H(a, b)$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a relative maximum;
(c) if $H(a, b) < 0$, then $f(a, b)$ is a saddle value.

If $H(a, b) = 0$, the theorem provides no information — anything can happen. Make up three examples $z = f(x, y)$ to illustrate how $f(a, b)$ could be a maximal z-value, a minimal z-value, or a saddle z-value in the ambiguous case $H(a, b) = 0$.

179. The function $H = f_{xx}f_{yy} - f_{xy}f_{yx}$ is sometimes called the *Hessian* of f. (By the way, in any situation where the second-derivative test is applicable, the partial derivatives f_{xy} and f_{yx} are equal.) What would a Hessian function for $w = f(x, y, z)$ look like?

180. The intuitive content of the Second-Derivative Test stems from the Maclaurin-series view of functions: If f is a "reasonable" function, then it is expected that

$$f(x, y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + \ldots$$

holds for all suitably small values of x and y, where the coefficients $a_{m,n}$ are partial derivatives of f, evaluated at $(0, 0)$ and divided by factorials; namely $a_{0,0} = f(0, 0)$, and

$$a_{m,n} = \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0, 0).$$

In particular, if $(0, 0)$ is a critical point for f, then

$$f(x, y) = f(0, 0) + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2 + \ldots,$$

and the quadratic terms are responsible for revealing how $z = f(x, y)$ relates to its horizontal tangent plane at $(0, 0, f(0, 0))$. For example, consider $f(x, y) = \cos x \cos y$.
(a) Use the Maclaurin expansion of f to analyze the surface $z = f(x, y)$ at $(0, 0, 1)$.
(b) Sketch several level curves of f.

181. Show that $f(x, y) = 2 + y^4 - 3x^2 - x^3 - 2y^2$ has six critical points. Use the Hessian function to classify each of them, then create a plausible system of level curves that is consistent with your data.

182. One reason why the Hessian test for critical points can fail is that a function can lack both linear and quadratic terms at the origin. For example, consider the cubic polynomial $f(x,y) = x^3 - 3xy^2$, whose only critical point is at the origin. Sketch plausible level curves for this function (it is not difficult), and thereby discover that there are actually many varieties of saddle point. This example is called a "monkey" saddle. Explain why.

183. Consider the interesting function

$$F(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } 0 < x^2+y^2; \\ 0, & \text{otherwise.} \end{cases}$$

The surface $z = F(x,y)$, in the vicinity of the origin, is shown below. Despite the awkward definition, which requires that $F(0,0)$ be stated as a special case, F is a continuous function. It is actually differentiable, for its partial derivatives exist and are continuous everywhere, even at the origin. Part (c) demonstrates that F has a strange property, however.

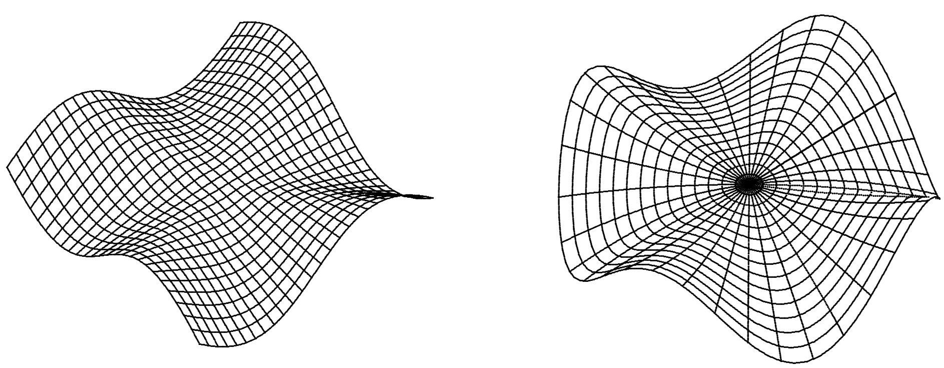
(a) Calculate F_x. This is a routine exercise at points other than the origin, and it requires evaluating a limit at the origin.

(b) Calculate F_y. This is a routine exercise at points other than the origin, and it requires evaluating a limit at the origin.

(c) Use your formulas for F_x and F_y to show that $F_{xy}(0,0) = -1$ and $F_{yx}(0,0) = 1$. Each result requires that you set up and evaluate a single limit.

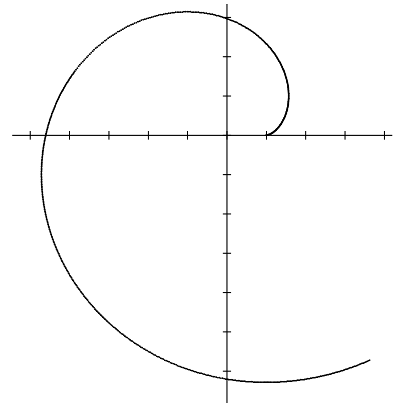
(d) The continuity of F and its first-order partial derivatives at the origin is most easily seen by expressing them in terms of polar coordinates. For example, $F(r,\theta) = \frac{1}{4}r^2 \sin 4\theta$. Verify this formula, and find similar formulas for F_x and F_y in terms of r and θ.

The diagram on the left uses the Cartesian grid for $-1 \le x \le 1$ and $-1 \le y \le 1$ to display the surface. The diagram on the right uses the polar grid for $0 \le r \le 1.25$ to display the same surface.



184. The diagram shows part of the spiral traced by the parametric equations

$$x = \cos t + t\sin t$$
$$y = \sin t - t\cos t.$$



Find the center of curvature at time $0 < t$.

185. The line through $(1, 11, 1)$ and $(7, 2, 13)$ does not intersect the line through $(-4, 4, 1)$ and $(8, 7, 4)$. Find the distance between these *skew* lines at their closest approach.

186. Consider the function $f(x, y) = (x^2 - y^2)(x^2 + y^2 - 1)$.
(a) Make a large sketch of the level curves of the surface $z = f(x, y)$.
(b) Calculate the gradient vector of f at $(0.8, 0.6)$, and add it to your diagram.
(c) What is the slope of the curve $f(x, y) = 12$ at $(2, 1)$?
(d) Find coordinates for all the critical points of f, and classify them.

187. Explain what it means for the vectors

$$\mathbf{u} = \frac{1}{7}[2, 3, 6] \qquad \mathbf{v} = \frac{1}{7}[3, -6, 2] \qquad \mathbf{w} = \frac{1}{7}[6, 2, -3]$$

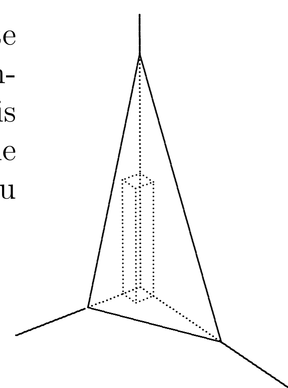
to form an *orthonormal* system, and find scalars a, b, and c so that $[10, 1, 2] = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

188. The parametric equations

$$x = \frac{6\cos kt}{2 + \cos kt} \qquad \text{and} \qquad y = \frac{6\sin kt}{2 + \cos kt}$$

describe the periodic motion of an object traveling along an ellipse in the xy-plane, where $k = \frac{1}{4}\pi$. Show that this path cannot represent the motion of a planet orbiting a sun at the origin. One piece of evidence is enough.

189. The plane $4x + 2y + z = 12$, together with the three coordinate planes, creates a triangular pyramid in the first octant. A rectangular box is inscribed in this pyramid, so that one of its corners is at the origin. Find the volume of the largest such box, and use the Second-Derivative Test to confirm the (local) maximum that you find.



190. An orthonormal system in $\mathbf{R}^4$ is formed by the vectors

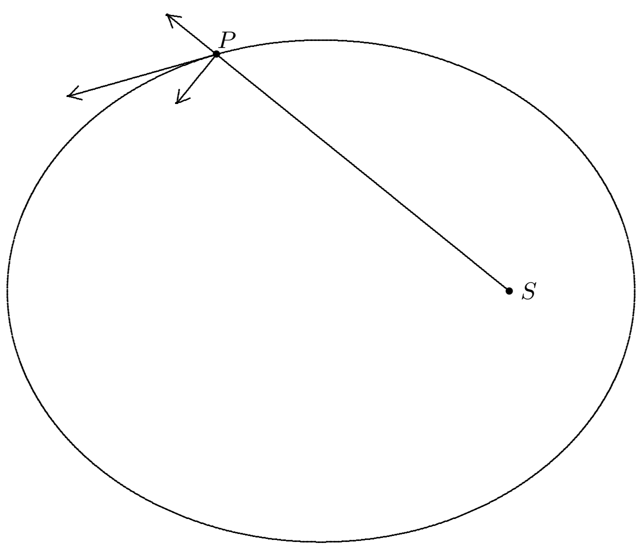
$$\mathbf{u}_1 = \frac{1}{5}[1,2,2,4], \quad \mathbf{u}_2 = \frac{1}{5}[2,-1,-4,2], \quad \mathbf{u}_3 = \frac{1}{5}[2,4,-1,-2], \quad \mathbf{u}_4 = \frac{1}{5}[4,-2,2,-1].$$

Find scalars a_1, a_2, a_3, and a_4 so that $[8,6,1,2] = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4$.

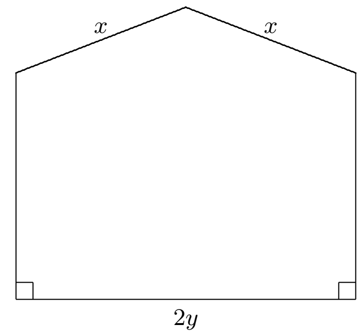
191. Consider the position vector $\mathbf{p} = [-3+5\cos kt, 4\sin kt]$, where k is a positive constant. (Notice that t is *not* the polar angle.)
(a) It so happens that $\mathbf{p}$ describes the position of a object that moves along an ellipse, one of whose focal points is at the origin. Find the eccentricity of the ellipse.
(b) No matter what value k has, this motion does *not* describe a planetary orbit. Confirm this statement by calculating at least one piece of numerical evidence.

192. You have seen this term two different methods of establishing "intrinsic" coordinate systems at points on parametrized curves — either by using $\mathbf{T}$ and $\mathbf{N}$ or by using $\mathbf{u}$ and $\mathbf{u}_\theta$ as a (moving) orthonormal basis. Given a polar curve $\mathbf{r} = r\mathbf{u}$, show how to express $\mathbf{T}$ in terms of $\mathbf{u}$ and $\mathbf{u}_\theta$. It is then easy to express $\mathbf{N}$ in terms of $\mathbf{u}$ and $\mathbf{u}_\theta$.

193. The diagram shows the ellipse traced by planet P as it travels around the Sun S. Three vectors based at P are also shown: the *unit* vectors $\mathbf{u}$ and $\mathbf{u}_\theta$, and the vector $\mathbf{r}'$.
(a) Calculate plausible values for r, r', and θ' at the instant when this picture was captured. Obtain all the numerical data you need by *measuring* the diagram.
(b) Use these values to draw the vectors $\mathbf{r}'$ at perihelion and aphelion.



194. A 2-meter length of wire is to be bent into the shape of a pentagon, which has an axis of reflective symmetry, and in which two adjacent angles are right, as shown in the diagram. The area of such a pentagon is



$$A(x,y) = 2y \cdot \left(1 - x - y + \frac{1}{2}\sqrt{x^2 - y^2}\right),$$

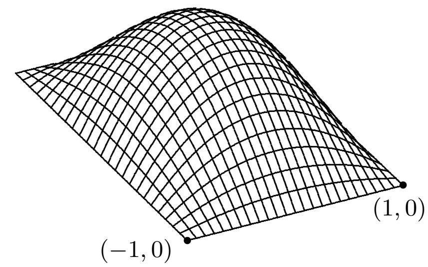
where x and y are marked in the diagram. Notice that the diagram and the equation make sense only for certain x and y-values.
(a) Describe the *domain* of $A(x,y)$ as a triangular subset of the xy-plane. Each of the three sides represents a special case of the allowed configurations. Explain.
(b) Show that the gradient of A is $\mathbf{0}$ at only one point that is within the triangular domain. Apply the second-derivative test to show that this point is a local maximum for A.
(c) Find the largest value of A along each of the three sides of its domain. At each such point, in what direction does the gradient of A point?
(d) Deduce the largest area that can be enclosed using the given 5-sided template.
(e) Consider the surface $z = A(x,y)$. Make a graphical representation of this surface that incorporates all of the preceding.

195. *Extreme values.* Given that f is a function that is defined and continuous for all points in a domain $\mathcal{D}$ that is *closed* and *bounded*, it can be proved that f attains absolute maximum and absolute minimum values on $\mathcal{D}$.
(a) The set $\mathcal{D}$ of points (x,y) for which $x^2 + y^2 < 1$ is an example of a set that is bounded but not closed. Explain the terminology, then find an example of a continuous function f defined on $\mathcal{D}$ that does not attain extreme values on $\mathcal{D}$.
(b) The set $\mathcal{D}$ of points (x,y) for which $1 \le x^2 + y^2$ is an example of a set that is closed but not bounded. Explain the terminology, then find an example of a continuous function f defined on $\mathcal{D}$ that does not attain extreme values on $\mathcal{D}$.

Suppose now that f is differentiable. It can be proved that the extreme values of f must occur at *critical* points of f, which are the points where the gradient of f is $\mathbf{0}$, along with all the points on the boundary of $\mathcal{D}$.
(c) What does this mean if the gradient of f is never $\mathbf{0}$?
(d) Let $\mathcal{D}$ be the points of the triangle defined by the simultaneous inequalities $0 \le x$, $0 \le y$, and $x + y \le 6$. What are the extreme values of $f(x,y) = xy(6 - x - y)$ on $\mathcal{D}$?

196. Find the volume of the largest rectangular box that can be inscribed in the ellipsoid $36x^2 + 9y^2 + 4z^2 = 36$, with the edges of the box being parallel to the coordinate axes.

197. The diagram shows $z = (1 - x^2)\sin y$ for the rectangular domain defined by $-1 \le x \le 1$ and $0 \le y \le \pi$. This surface and the plane $z = 0$ enclose a region $\mathcal{R}$. It is possible to find the volume of $\mathcal{R}$ by integration:



(a) Notice first that $\mathcal{R}$ can be sliced neatly into sections by cutting planes that are perpendicular to the y-axis — one for each value of y between 0 and π, inclusive. The area $A(y)$ of the slice determined by a specific value of y can be found using ordinary integration. Calculate it.
(b) Use the slice-area function $A(y)$ to find the volume of $\mathcal{R}$.
(c) Notice also that $\mathcal{R}$ can be sliced into sections by cutting planes that are perpendicular to the x-axis — one for each value of x between -1 and 1. As in (a), use ordinary integration to find the area $B(x)$ of the slice determined by a specific value of x.
(d) Integrate $B(x)$ to find the volume of $\mathcal{R}$.

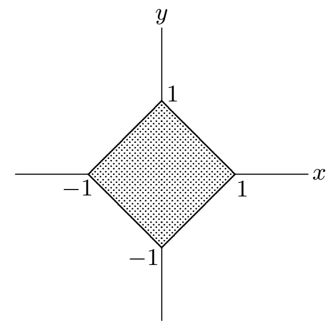
198. The preceding problem illustrates how a problem can be solved using *double integration.* Justify the terminology (it does not mean that the problem was actually solved twice). Notice that the example was made especially simple because the limits on the integrals were constant — the limits on the integral used to find $A(y)$ did not depend on y, nor did the limits on the integral used to find $B(x)$ depend on x. The method of using cross-sections to find volumes can be adapted to other situations, however. For example, consider the region $\mathcal{R}$ enclosed by the surface $z = xy(6 - x - y)$ and the plane $z = 0$ for $0 \le x$, $0 \le y$, and $x + y \le 6$. Find the volume of $\mathcal{R}$.

199. Let $V(x, y) = 1 - x^2 - y^2$ be interpreted as the speed (cm/sec) of fluid that is flowing through point (x, y) in a pipe whose cross section is the unit disk $x^2 + y^2 < 1$. Assume that the flow is the same through every cross-section of the pipe. Notice that the flow is most rapid at the center of the pipe, and is rather sluggish near the boundary.

The volume of fluid that passes each second through any *small* cross-sectional box whose area is $\Delta A = \Delta x \Delta y$ is approximately $V(x, y)\Delta x \Delta y$, where (x, y) is a representative point in the small box.
(a) Using an integral with respect to y, combine these approximations to obtain an approximate value for the volume of fluid that flows each second through a strip of width Δx that is parallel to the y-axis. The result will depend on the value of x that represents the position of this strip.
(b) Use integration with respect to x to show that the volume of fluid that leaves the pipe (through the cross-section at the end) each second is $\pi/2 \approx 1.57$ cc.

200. (Continuation) What is the average speed of the fluid?

201. Water is flowing through the square pipe whose cross-section is shown in the diagram. The speed of the flow at point (x, y) is $f(x, y) = 1 - |x| - |y|$ cm per second. At what rate, in cc per second, is water flowing through the pipe?



202. (Continuation) Notice that the speeds of individual water molecules vary from 0 (at the boundary) to 1 (at the center). What is the *average* speed of the water as it flows through the pipe? Explain your choice.

203. Consider the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$. Describe their points of intersection. In particular, how would this configuration of points look to an observer stationed at $(100, 0, 0)$? How about an observer stationed at $(0, 0, 100)$?

204. Consider the region of space that is common to the two solid cylinders $x^2 + z^2 \le 1$ and $y^2 + z^2 \le 1$. Use the cross-sectional approach to find its volume.

205. The function $f(x, y) = y^2 + 4y - x^2 y$ is defined for all points in the circular domain $\mathcal{R} = \{(x, y) \,|\, x^2 + y^2 \le 9\}$. What is the range of values of f on $\mathcal{R}$? In particular, what are the extreme values of f, and where are they attained?

206. Given a point inside the unit circle, the distance to the origin is some number between 0 and 1. What is the *average* of all these distances? It should also be a number between 0 and 1. Justify your approach.

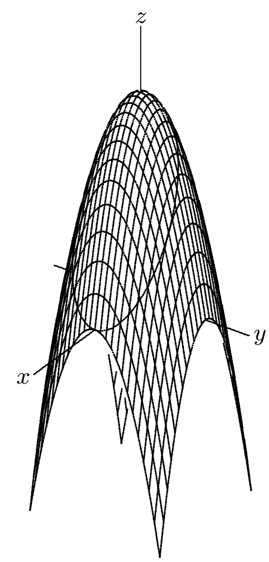
207. Consider the triangle $\mathcal{T}$ on the xy-plane formed by points $A = (0, 0, 0)$, $B = (1, 0, 0)$, and $C = (1, 1, 0)$. The surface $z = y \cos\left(\frac{\pi}{2} x^3\right)$, along with $\mathcal{T}$ and the plane $y = x$, enclose a region of space. Make a sketch of this region, then use the cross-sectional method to find its volume. You might notice that it makes a difference whether you begin by slicing the region perpendicular to the x-axis (the $dy\,dx$ approach) or by slicing perpendicular to the y-axis (the $dx\,dy$ approach).

208. The lines $(x, y, z) = (-1 + 4t, 3 - t, 7 - t)$ and $(x, y, z) = (1 - 2u, -2 - 4u, 20 + 5u)$ are *skew*, which means that they are not parallel and do not intersect. Find the smallest distance that separates a point on one line from a point on the other line.

209. Given $f(x, y)$, let $F(t) = f(0.8t, 0.6t)$. The coefficients a_n of the Maclaurin expansion $F(t) = a_0 + a_1 t + a_2 t^2 + \ldots$ can be expressed in terms of f and its partial derivatives f_x, f_y, f_{xx}, f_{xy}, $\ldots$. For example, $a_1 = F'(0)$ becomes $0.8 f_x(0, 0) + 0.6 f_y(0, 0)$. Justify this equation, then write similar expressions for a_0, a_2, and a_3. Look for patterns.

210. Suppose that the temperature at point (x, y) of a metal plate is $T(x, y) = 100e^{-x} \sin y$, for $0 \le x \le 1$ and $0 \le y \le \pi$. The temperatures in this plate therefore range between 0 and 100 degrees, inclusive. What is the average of all these temperatures?

211. The paraboloid $z = 9 - x^2 - y^2$ is cut by the plane $z = 6 - 2x$. The intersection curve is an ellipse, most of which is showing in the diagram. Use double integration to find the volume of the region that is enclosed by these two surfaces. In other words, the region is above the plane and below the paraboloid.



212. The function $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ is defined and differentiable at every point in the plane, and the only critical point of f is a local maximum. The Second-Derivative Test can be used to check this, but it is not needed. It so happens that this solitary critical point is actually a *global* maximum. Confirm that the preceding statements are true, then ponder the question: Is it true that an everywhere-differentiable function that has only *one* critical point — that point being a local maximum — *must* have a global maximum?

213. Consider the plane $z = 5 - 2x - 2y$ and the points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (1, 1, 0)$, $D = (0, 1, 0)$, $P = (0, 0, 5)$, $Q = (1, 0, 3)$, $R = (1, 1, 1)$, and $S = (0, 1, 3)$. Notice that $PQRS$ is a quadrilateral that lies on the plane, and that $ABCD$ is its projection onto the xy-plane. Compare the areas of these quadrilaterals. Then choose a different example of the same sort — a quadrilateral on the plane $z = 5 - 2x - 2y$ and its projection onto the xy-plane — and compare their areas. Look for a pattern and explain it.

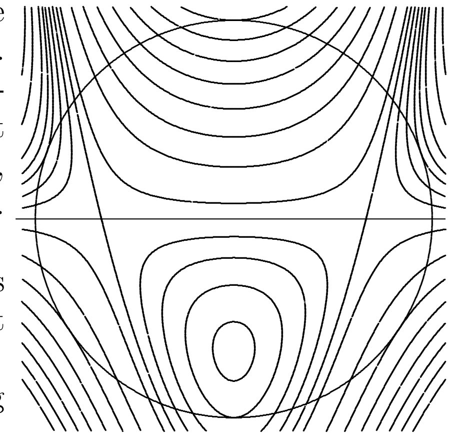
214. Evaluate the double integral $\displaystyle\int_0^1 \int_x^1 \cos\left(y^2\right)\, dy\, dx$ without using a calculator. You need to describe the domain of the integration in a way that is different from the given description. This is called *reversing the order of integration.*

215. *Fubini's Theorem* states that

$$\int_c^d \int_a^b f(x, y)\, dx\, dy \;=\; \int_a^b \int_c^d f(x, y)\, dy\, dx$$

is true whenever f is a function that is continuous at all points in the rectangle $a \le x \le b$ and $c \le y \le d$. Despite the intuitive content of this statement, a proof is not easy, and this will be left for a later course. It suffices to do examples that illustrate its non-trivial content. Verify the conclusion of the theorem using $f(x, y) = x \sin(xy)$ and the rectangle $1 \le x \le 2$ and $0 \le y \le \pi$.

216. The diagram shows several level curves for the function $f(x,y) = y^2 + 4y - x^2y$. It also shows the circle $x^2+y^2 = 9$, which is tangent in *six* places to level curves. To find the complete range of values of f when it is restricted to the circular domain $x^2 + y^2 \le 9$, you must examine the value of $f(x, y)$ at each of these six points, and at each of the three critical points inside the circle. Explain why.



(a) The gradient of f is $[-2xy, 2y + 4 - x^2]$. How does this vector relate to the level curves of f? How does it relate to the circle $x^2 + y^2 = 9$?
(b) The six points of tangency can be found by looking for points where the gradient of f is parallel to the gradient of $g(x,y) = x^2 + y^2$. Explain this reasoning, and then work on the resulting equation until you make it look like $3y^2 + 2y = 5$.
(c) To complete the critical-point analysis, you should examine the nine points $(-2, 0)$, $(2, 0)$, $(0, -2)$, $(0, 3)$, $(0, -3)$, $(\sqrt{8}\,, 1)$, $(-\sqrt{8}\,, 1)$, $(-\sqrt{56}/3, -5/3)$, and $(\sqrt{56}/3, -5/3)$, all of which are conspicuous in the diagram. Identify each of them.

217. *Constraints.* The boundary-curve part of the preceding question illustrates how one uses gradients to find extreme values of a function whose domain has been restricted by means of another function. The typical problem takes the form "Find the extreme values of $f(x,y)$, given that $g(x,y) = k$", and the method is to look for points where ∇f is parallel to ∇g. Try out this method on a simple and familiar example: Find dimensions r and h for a cylindrical can whose volume is 1000 cc (this is the constraint) and whose total surface area is minimal. In your solution, do not immediately solve for one variable in terms of the other (the usual approach), but use gradients instead; this treats both variables equally.

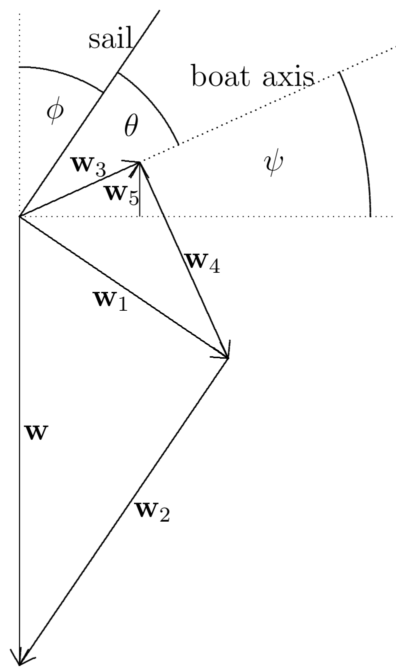
218. A constrained minimum: Find the point on the plane $2x+6y+3z = 98$ that minimizes the function $f(x,y,z) = x^2 + y^2 + z^2$.

219. Another constrained minimum: Find the point on the paraboloid $x^2+y^2+z = 16$ that is closest to the point $(6, 9, 4)$. It is convenient to use an auxiliary variable t in the solution to stand for the multiplier that relates the two gradients. Thus let $\nabla f = t\nabla g$, express x, y, and z in terms of this *Lagrange multiplier*, and solve the resulting cubic equation.

220. Verify that the point $P = (2, 6, 3)$ is on the sphere $x^2 + y^2 + z^2 = 49$. Consider a *small* piece $\mathcal{S}$ of the spherical surface that includes P, and let $\mathcal{R}$ be the projection of $\mathcal{S}$ onto the xy-plane.
(a) Explain why the area of $\mathcal{R}$ is approximately $3/7$ times the area of $\mathcal{S}$, and why the approximation gets better and better as the dimensions of $\mathcal{S}$ decrease to zero.
(b) By considering the angle between two vectors, explain where the ratio $3/7$ comes from.
(c) What would the ratio have been if $P = (0, 0, 7)$ had been selected instead?

221. Explain why the vector $[-2x, 2y, 1]$ is perpendicular to the saddle surface $z = x^2 - y^2$ at (x, y, z). Consider that part $\mathcal{S}$ of the surface that is bounded by $-2 \le x \le 2$ and $-2 \le y \le 2$. Explain why the area of $\mathcal{S}$ is exactly $\displaystyle\int_{-2}^{2}\int_{-2}^{2}\sqrt{4x^2+4y^2+1}\,dx\,dy$.

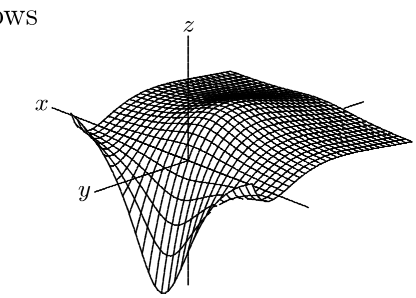
222. *Tacking into the wind.* The diagram shows how the velocity $\mathbf{w}$ of a northerly wind (blowing down the page) is resolved into components that describe the effect of the wind on a sailboat trying to move *up* the page. The angle made by the sail and the direction of the boat is θ, the angle made by the sail and the (northerly) direction from which the wind comes is ϕ, and the angle between the direction of the boat and the easterly direction is ψ — in other words, ψ makes $\theta + \phi + \psi = \frac{1}{2}\pi$ true.



(a) Let $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is the component of $\mathbf{w}$ that is perpendicular to the sail. Explain why $|\mathbf{w}_1| = |\mathbf{w}|\sin\phi$.
(b) Let $\mathbf{w}_1 = \mathbf{w}_3 + \mathbf{w}_4$, where $\mathbf{w}_3$ is the vector projection of $\mathbf{w}_1$ on the direction of the boat. Explain why $|\mathbf{w}_3| = |\mathbf{w}|\sin\theta\sin\phi$.
(c) Let $\mathbf{w}_3 = \mathbf{w}_5 + \mathbf{w}_6$, where $\mathbf{w}_5$ is the vector projection of $\mathbf{w}_3$ on the direction of the wind. Explain why $|\mathbf{w}_5| = |\mathbf{w}|\sin\theta\sin\phi\sin\psi$.
(d) Maximize the product $\sin\theta\sin\phi\sin\psi$, subject to the constraint $\theta + \phi + \psi = \frac{1}{2}\pi$. In other words, tell the boat's crew how to maximize their rate of progress upwind.
(e) What percentage of the force of the wind is used to propel the boat upwind?

223. There are many planes that contain the point $P = (2, 6, 3)$ and that have positive intercepts with all three coordinate axes. Any such plane, together with the planes $x = 0$, $y = 0$, and $z = 0$, defines a triangular pyramid in the first octant. The goal is to find the plane containing P that *minimizes* the volume of this pyramid. For example, the plane $2x + y + 2z = 16$ contains P, and the volume it cuts off is $512/3$ (not minimal). Notice that the unknowns of this problem are the coefficients of the equation (not x, y, and z).

224. Let $f(x,y) = \left(\dfrac{x^2e^y - 1}{x^2+1} - 1\right)ye^y$. The diagram shows part of the surface $z = f(x, y)$. Show that
(a) $f(0, -1) = 2/e$ is a local maximum;
(b) there are no other critical points for f; and
(c) *all* numbers are in the range of $f(x, y)$. Hmm ...



225. Given that $\mathbf{w}$ is the vector projection of $\mathbf{u}$ onto $\mathbf{v}$, why is $\mathbf{u} - \mathbf{w}$ perpendicular to $\mathbf{v}$?

226. Suppose that f and g are differentiable, and that P satisfies the constraint $g(P) = k$. How can one tell whether P produces an extreme value of f? A necessary condition is that the gradient vectors ∇f and ∇g be parallel. To prove this, let $\mathbf{w}_1$ be the projection of ∇f onto ∇g, and let $\mathbf{w}_2 = \nabla f - \mathbf{w}_1$.
(a) If $\mathbf{w}_2$ is the zero vector, there is nothing more to prove. Explain, then assume that $\mathbf{w}_2$ is nonzero.
(b) Explain why $\mathbf{w}_2$ is tangent to the constraint (surface or curve) $g = k$.
(c) Because $f(P)$ is an extreme value of f constrained by $g = k$, it is necessary that the directional derivative $D_{\mathbf{w}_2} f(P)$ be zero. Explain.
(d) Recall that $D_{\mathbf{w}_2} f = \mathbf{w}_2 \bullet \nabla f$. Thus $0 = \mathbf{w}_2 \bullet (\mathbf{w}_1 + \mathbf{w}_2)$. Explain.
(e) Conclude that $\mathbf{w}_2 = \mathbf{0}$, as desired.

227. Evaluate the integral $\displaystyle\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\, dy\, dx$. Then reverse the order of integration and evaluate again. Explain why you should not be surprised by the result.

228. In setting up a double integral, it is customary to tile the domain of integration using little rectangles whose areas are $\Delta x \Delta y$. In some situations, however, it is better to use small tiles whose areas can be described as $r \Delta r \Delta \theta$. Sketch such a tile, and explain the formula for its area. In what situations would such tiles be useful?

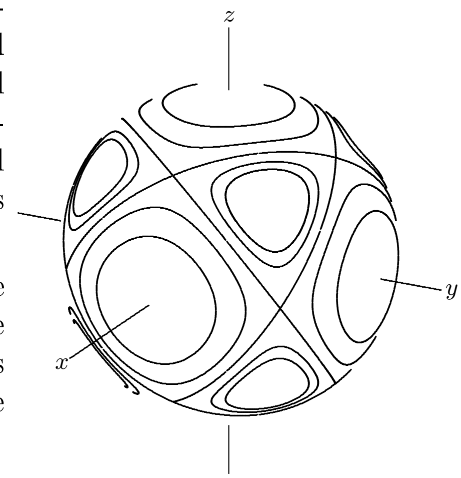
229. Suppose that f is a function for which the second-order partial derivative f_{yx} is continuous. Given the four values $f(0,0) = m$, $f(1,0) = n$, $f(1,1) = p$, and $f(0,1) = q$, evaluate the integral $\displaystyle\int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x \partial y}\, dx\, dy$, in terms of the numbers m, n, p, and q. Yes, the two notations for partial derivatives are consistent.

230. *Lagrange multipliers.* Here is how Lagrange would have found extreme values of $f(x, y, z)$, subject to the constraint $g(x, y, z) = 0$: Form the auxiliary function

$$h(x, y, z, t) = f(x, y, z) - t \cdot g(x, y, z).$$

Find all solutions to the four simultaneous equations $h_x = 0$, $h_y = 0$, $h_z = 0$, and $h_t = 0$. How does this compare with the gradient-vector approach?

231. Suppose that $T(x, y, z) = x^4 + y^4 + z^4$ is the temperature at point (x, y, z), and let $\mathcal{S}$ be the spherical surface $x^2 + y^2 + z^2 = 3$.
(a) Describe the isothermal surfaces of T.
(b) Find all the points on $\mathcal{S}$ where T equals 9. These are points of extreme temperature on $\mathcal{S}$. Are they relative maxima or minima?
(c) Find all 26 critical points for the temperature function T restricted to $\mathcal{S}$.



232. The diagram shows a few isothermals for the temperature function $T(x, y, z) = x^4 + y^4 + z^4$, restricted to the sphere $x^2 + y^2 + z^2 = 3$. Explain the symmetrical arrangement of the six points where the maximal temperature 9 is attained, the eight points where the minimal temperature 3 is attained, and the twelve saddle points that all produce the temperature $T = 4.5$.

It is interesting that the isothermal $T = 4.5$ intersects the sphere in four great circles (all of which can be seen in the diagram). The twelve saddle points are the intersections of these great circles. Explain why there can be no more than twelve intersections of four great circles.

233. (Continuation) At any point (x, y, z), you can calculate the cross product of the vectors $[4x^3, 4y^3, 4z^3]$ and $[2x, 2y, 2z]$. Do so. What is the significance of this vector when (x, y, z) is a point of the sphere shown above?

234. Explain why $2\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$ equals the surface area of the unit sphere. Explain why this is an improper integral, then describe how to deal with it properly.

235. (Continuation) In problems like the preceding, a different coordinate system works better. Show how to re-express the problem using polar variables r and θ instead of x and y. This requires that you replace the differential element of area $dx\,dy$ by something polar, and that you put new limits on the integrals. Evaluate the resulting double integral. You should not need your calculator for this version of the question.

236. Convert $\int_0^1\int_y^1 x\,dx\,dy$ into polar form. Evaluate both versions of the double integral and interpret the result.

237. If a random point were chosen in the square defined by $0 \le x \le 1$ and $0 \le y \le 1$, its distance from the origin would be somewhere between 0 and $\sqrt{2}$. What is the average of all these distances?

238. What is the range of values of $f(x, y, z) = xyz$, if (x, y, z) is restricted to points of the unit sphere $x^2 + y^2 + z^2 = 1$?

239. Find the volume of the solid region enclosed by the xy-plane, the cylinders $r = 1$ and $r = 2$, the planes $\theta = 0$ and $\theta = \pi/2$, and the plane $z = x + y$.

240. Suppose that f is a function of x and y, and that both second-order partial derivatives f_{xy} and f_{yx} are continuous on a rectangle $a \le x \le b$ and $c \le y \le d$. You have shown that

$$\int_c^d \int_a^b f_{yx}\,dx\,dy = f(b,d) + f(a,c) - f(b,c) - f(a,d)$$

and

$$\int_a^b \int_c^d f_{xy}\,dy\,dx = f(b,d) + f(a,c) - f(b,c) - f(a,d).$$

(a) Explain why

$$\int_c^d \int_a^b (f_{yx} - f_{xy})\;dx\,dy = 0.$$

(b) Suppose that $f_{xy}(t,u) < f_{yx}(t,u)$ at some point (t,u). By considering a *suitably small* rectangle, derive a contradiction from (a). Draw the conclusion that $f_{xy} = f_{yx}$ whenever both functions are continuous.

241. Discuss the definition

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \lim_{a\to\infty} \int_0^a \int_0^a f(x,y)\,dx\,dy\,.$$

242. Explain why

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right).$$

243. Explain why

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\,r\,dr\,d\theta.$$

244. Explain why

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}\,.$$

First proved by Laplace, this is a significant result for statistical work.

245. Write $\mathbf{v} = [6, 2, 3]$ as a sum $\mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{u} = [2, 1, 2]$, and $\mathbf{v}_2$ is perpendicular to $\mathbf{u}$. Check your arithmetic.

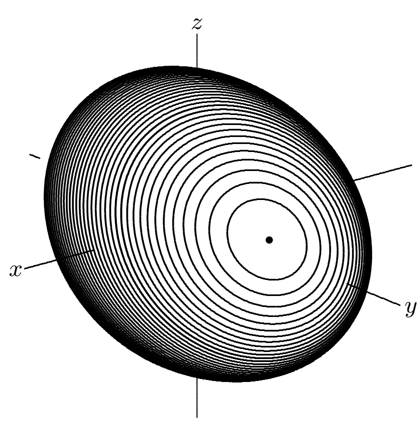
246. Reverse the order of integration in $\displaystyle\int_0^3 \int_{9-3x}^{9-x^2} f(x, y)\, dy\, dx$. In other words, rewrite the integral using $dx\, dy$ as the area differential.

247. Let $\mathcal{P}$ be the surface $x^2 + 4y^2 + z = 16$, called a *paraboloid*. Set up a double integral whose value is the *area* of that part of $\mathcal{P}$ that lies above the plane $z = 0$. You do *not* need to evaluate your integral.

248. Let $\mathcal{S}$ be the spherical surface $x^2 + y^2 + z^2 = 19$. The preceding paraboloid $\mathcal{P}$ intersects the sphere $\mathcal{S}$ along a differentiable curve that goes through the point $B = (3, 1, 3)$. Do not attempt to find an equation for this curve, but do find a vector that is tangent to it at B.

249. Convert the double integral $\displaystyle\int_1^2 \int_{-x}^{x} (x^2 + y^2) \arctan \frac{y}{x}\, dy\, dx$ into an equivalent form expressed in terms of polar coordinates r and θ. Do not try to evaluate either integral.

250. Let $f(x, y, z) = 2x + 2y + z$. Find the range of values of f when (x, y, z) is restricted to lie on the surface $3x^2 + y^2 + 2z^2 = 210$.



251. For exactly what values of k does the paraboloid $x^2 + 4y^2 + z = k$ have points in common with the sphere $x^2 + y^2 + z^2 = 19$?

252. Let $p(x, y) = y^2 + (xy - 1)^2$.
(a) Show that p has exactly one critical point.
(b) Classify the unique critical point.
(c) Show that the values of p include all positive numbers.
(d) Show that p does not have a globally minimal value.

253. Consider that part of the saddle surface $z = x^2 - y^2$ that lies above the disk $x^2 + y^2 \le 6$. Show that its area is a rational multiple of π.

254. Let $f(0) = 2$, and for nonzero values of x, let $f(x) = \dfrac{e^{-x} - e^{-3x}}{x}$.
(a) Show that f is differentiable at $x = 0$.
(b) Evaluate the improper integral $\int_0^\infty f(x)\,dx$. This can be done by the "trick" of creating an appropriate double integral and reversing the order of integration.

255. Evaluate $\displaystyle\int_{-1}^{2}\int_{2-x}^{4-x^2} dy\,dx$. Explain why your answer can be considered as an area or as a volume.

256. Is it feasible to reverse the order of integration in the preceding?

257. Evaluate $\displaystyle\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$. Explain why your answer can be considered as an area or as a volume.

258. Is it feasible to reverse the order of integration in the preceding?

259. Let $\mathcal{R}$ be the triangular region whose vertices are $(0,0)$, $(1,0)$, and $(1,\sqrt{3})$. Let $f(x,y) = \dfrac{1}{(1+x^2+y^2)^2}$. Evaluate the integral of f over $\mathcal{R}$.

260. Consider that part of the "monkey" saddle $z = \frac{1}{3}x^3 - xy^2 + 2$ that lies above the disk $x^2 + y^2 \le 3$ in the xy-plane. Find its area.

261. What is the average distance from $(0,0,1)$ to an arbitrary point on the unit sphere $x^2 + y^2 + z^2 = 1$? The answer is a rational number.

262. The familiar equations $x = r\cos\theta$, $y = r\sin\theta$ can be thought of as a *mapping* from the $r\theta$-plane to the xy-plane. In other words, $p(r,\theta) = (r\cos\theta, r\sin\theta)$ is a function of the type $\mathbf{R}^2 \to \mathbf{R}^2$. Point by point, p transforms regions of the $r\theta$-plane onto regions of the xy-plane. In particular, consider the rectangle defined by $2 \le r \le 2.1$ and $1 \le \theta \le 1.2$. What is its image in the xy-plane? How do the areas of these two regions compare?

263. (Continuation) The derivative of p at $(2,1)$, which could be denoted $p'(2,1)$, is a 2×2 matrix, and its determinant is an interesting number. Explain these statements. It may help to know that these determinant matrices are usually denoted $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

264. Find the smallest volume in the first octant that can be cut off by a plane through the point $P = (2, 6, 3)$.

Method I. The plane can be described by an equation $ax + by + cz = 2a + 6b + 3c$, where a, b, and c are positive. The plane intercepts the x-axis at $(2a + 6b + 3c)/a$, the y-axis at $(2a + 6b + 3c)/b$, and the z-axis at $(2a + 6b + 3c)/c$. Thus the volume to be minimized is

$$V(a, b, c) = \frac{(2a + 6b + 3c)^3}{6abc} .$$

Begin by calculating the gradient:

$$\nabla V = \frac{(2a + 6b + 3c)^2}{6a^2b^2c^2} \left[(4a - 6b - 3c)bc, (12b - 2a - 3c)ac, (6c - 2a - 6b)ab\right]$$

In order for this to be $\mathbf{0}$, it is necessary that $4a - 6b - 3c = 0$ and $12b - 2a - 3c = 0$ and $6c - 2a - 6b = 0$. The third equation is redundant, and any multiple of $a = 3$, $b = 1$, and $c = 2$ will work. The minimal volume is $V(3, 1, 2) = 162$.

Method II. Let p, q, and r be the intercepts of the requested plane with the x, y, and z axes, respectively. The plane can thus be described by the equation $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$, and the volume to be minimized is $f(p, q, r) = \frac{1}{6}pqr$. Because the plane must contain $(2, 6, 3)$, the constraint $g(p, q, r) = \frac{2}{p} + \frac{6}{q} + \frac{3}{r} = 1$ applies. The Lagrange method tells you to look for (p, q, r) that make

$$\nabla f = \frac{1}{6}[qr, pr, pq] \quad \text{a multiple of} \quad \nabla g = \left[-\frac{2}{p^2}, -\frac{6}{q^2}, -\frac{3}{r^2}\right] .$$

Because pqr is nonzero, this implies that

$$\frac{qr}{6} \cdot \frac{p^2}{-2} = \frac{pr}{6} \cdot \frac{q^2}{-6} = \frac{pq}{6} \cdot \frac{r^2}{-3},$$

which leads to $p = 2t$, $q = 6t$, and $r = 3t$. Substitute into $g(p, q, r) = 1$ to find that $t = 3$. Thus $p = 6$, $q = 18$, $r = 9$, and the minimal volume is $f(6, 18, 9) = 162$.

265. Does either of these two approaches actually show that *every* plane passing through $(2, 6, 3)$ cuts off a volume that is at least as large as 162?

266. Let $\mathcal{R}$ be the sector defined by $0 \le r \le 1$ and $0 \le \theta \le \beta$. Find coordinates (polar and Cartesian) for the centroid of $\mathcal{R}$. Your answer will of course depend on β. In particular, the centroid should be at the origin when $\beta = 2\pi$, and its r-coordinate should be very close to $2/3$ when β is small. (By the way, the average r-value for any sector is $2/3$, but this is not what the question is asking for.)

267. Consider the linear mapping $g : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $x = 3u + v$ and $y = u + 2v$. In other words, $g(u, v) = (3u + v, u + 2v)$. Point by point, g transforms regions of the uv-plane onto regions of the xy-plane. Select any uv-rectangle and calculate its g-image (which is a simple geometric shape). After you compare the area of the image with the area of the rectangle, calculate the determinant of $g'(0, 0)$, which is the 2×2 matrix $\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$.

268. Consider the function $f(u, v) = (u^2 - v^2, 2uv)$. Apply it to the rectangle $\mathcal{R}$ defined by $1 \le u \le 1.5$ and $1 \le v \le 1.5$. Show that the image "quadrilateral" $\mathcal{Q}$ is enclosed by four parabolic arcs. First estimate the area of $\mathcal{Q}$, then calculate it exactly. What is the ratio of this area to the area of $\mathcal{R}$?

269. (Continuation) Apply f to the rectangle $\mathcal{R}$ defined by $1 \le u \le 1.1$ and $1 \le v \le 1.1$. The image $\mathcal{Q}$ is enclosed by four parabolic arcs. Make a detailed sketch of $\mathcal{Q}$. Calculate the matrix $f'(1, 1)$, and then find its determinant. You should expect the area of $\mathcal{Q}$ to be approximately 8 times the area of $\mathcal{R}$. Explain why.

270. (Continuation) Apply the function $g(h, k) = (2h - 2k, 2h + 2k)$ to the rectangle defined by $0 \le h \le 0.1$ and $0 \le k \le 0.1$. Compare the result with the $\mathcal{Q}$ calculated in the preceding item. Then explain what the matrix $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ reveals about the mapping f in the vicinity of $(u, v) = (1, 1)$.

271. In general, given a mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$, its derivative is a 2×2 matrix-valued function that provides a *local multiplier* at each point of the domain of f. Each such matrix describes how suitable domain rectangles are transformed into image quadrilaterals, and its determinant is a multiplier that converts (approximately) the rectangular areas into the quadrilateral areas. Explain the words "local" and "suitable", and make use of the limit concept in your answer. It is customary to refer to either the matrix f' or its determinant as *the Jacobian of f*.

272. Explain why each row of a Jacobian matrix is the gradient of a certain function.

273. Justify the equation $\dfrac{\partial(x, y)}{\partial(u, v)}\, du\, dv = dx\, dy$.

274. Let $\mathcal{R}$ be the rectangular region defined by $0 \le u \le 2$ and $1 \le v \le 2$. Let $\mathcal{Q}$ be the region obtained by applying the mapping $(x, y) = (u^2 - v^2, 2uv)$ to $\mathcal{R}$.
(a) Sketch the four-sided region $\mathcal{Q}$.
(b) Find the area of $\mathcal{Q}$.

275. Given a circle of unit radius, find the average distance from a point on the circle to an arbitrary point inside. (*Hint*: consider the circle $r = 2\cos\theta$).

276. Let $\mathcal{S}$ be defined by $z = 1 + x^2 + y^2$ for $x^2 + y^2 \le 9$, making $\mathcal{S}$ part of a paraboloid. Find the area of $\mathcal{S}$.

277. The function $f(x, y) = (x, y, 1 + x^2 + y^2)$ maps $\mathbf{R}^2 \to \mathbf{R}^3$. Write down a matrix that deserves to be called $f'(2, 1)$.

278. What is the area of the parallelogram $ABCD$ whose vertices are $A = (2, 1, 6)$ and
(a) $B = (3, 1, 10)$, $C = (3, 2, 12)$, and $D = (2, 2, 8)$?
(b) $B = (2 + h, 1, 6 + 4h)$, $C = (2 + h, 1 + k, 6 + 4h + 2k)$, and $D = (2, 1 + k, 6 + 2k)$?

279. Consider the linear function $L(u, v) = (2 + u, 1 + v, 6 + 4u + 2v)$. When L is applied to a polygonal region $\mathcal{R}$, the result is a planar polygon $\mathcal{Q}$ in 3-dimensional space. What number is obtained when the area of $\mathcal{Q}$ is divided by the area of $\mathcal{R}$?

280. Find coordinates (x, y) for the centroid of the region $\mathcal{Q}$ of item 1 above.

281. The appearance of the integral $\displaystyle\int_1^4 \int_{1/x}^{4/x} \frac{xy}{1 + x^2y^2}\, dy\, dx$ suggests that it would be helpful if xy were a single variable. With this in mind, consider the transformation of coordinates $(x, y) = (u, v/u)$.
(a) Sketch the given region of integration in the xy-plane.
(b) Show that this region is the image of a square region in the uv-plane.
(c) Evaluate the given integral by making the indicated change of variables.

The purpose of the next three questions is to evaluate the improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$.

282. Assume that $0 < x$, and show that $\displaystyle\int_0^\infty e^{-xy}\, dy = \frac{1}{x}$.

283. Assume that $0 < y$, and show that $\displaystyle\int_0^\infty e^{-xy} \sin x\, dx = \frac{1}{1 + y^2}$. Integration by parts is an effective approach.

284. Use an improper double integral to help you deduce the value of $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$.

285. If x and y are nonnegative numbers, then $\sqrt{xy} \le \frac{1}{2}(x+y)$, and equality occurs only if $x = y$. This result is called the *inequality of the arithmetic and geometric means*, or *AM-GM* for short. It is a simple workout in algebra.

286. If x, y, z, and t are nonnegative numbers, then $\sqrt[4]{xyzt} \le \frac{1}{4}(x+y+z+t)$, and equality occurs only if $x = y = z = t$. This result is another instance of *AM-GM*. You can prove this version by applying the preceding result to $\sqrt{xy}$ and $\sqrt{zt}$. Try it.

287. State another instance of *AM-GM* — one that you can prove — and then prove it. If there are a lot of variables, you may want to subscript them.

288. If x, y, and z are nonnegative numbers, then $\sqrt[3]{xyz} \le \frac{1}{3}(x+y+z)$, and equality occurs only if $x = y = z$. This three-variable version of *AM-GM* can be proved by appealing to the four-variable version, by setting $t = \sqrt[3]{xyz}$. Show how.

289. As an application of the three-variable *AM-GM*, finish the job of showing that 162 is the smallest volume in the first octant that can be cut off by a plane through the point $P = (2, 6, 3)$. In other words, show that $162 \le \dfrac{(2a+6b+3c)^3}{6abc}$ for any positive values a, b, and c, with equality only when $a = 3$, $b = 1$, and $c = 2$. Recall that the equation $ax + by + cz = 2a + 6b + 3c$ describes all the planes that go through P.

290. Given a function f that is differentiable at a, the tangent line is distinguished from any other line $y = m(x-a) + f(a)$ that goes through $(a, f(a))$ by the following property: Only if $m = f'(a)$ is it true that the difference between $f(x)$ and $m(x-a)+f(a)$ approaches 0 *faster than x approaches a.* This means that $0 = \displaystyle\lim_{x\to a} \frac{f(x) - m(x-a) - f(a)}{x-a}$. Verify the truth of this statement.

291. (Continuation) In this sense, the function $L(x) = f(a) + f'(a)(x-a)$ stands out as the *best* linear approximation to f at $x = a$. This definition of differentiability can be applied to other functions. For example, $f(x, y)$ is differentiable at (a, b) if there is a linear function $L(x, y)$ with the property that the difference between $L(x, y)$ and $f(x, y)$ approaches 0 faster than (x, y) approaches (a, b), meaning that $0 = \displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}}$. A less cumbersome description of differentiability at $\mathbf{p}$, which in fact applies to examples of *any* dimension, is $0 = \displaystyle\lim_{|\mathbf{h}|\to 0} \frac{f(\mathbf{p}+\mathbf{h}) - L(\mathbf{p}+\mathbf{h})}{|\mathbf{h}|}$, where $\mathbf{h}$ denotes the displacement vector. In this view, $f'(\mathbf{p})$ is the matrix for which $L(\mathbf{p}+\mathbf{h}) = f(\mathbf{p}) + f'(\mathbf{p})\mathbf{h}$.
(a) What are the components of matrix $f'(\mathbf{p})$ in the $\mathbf{R}^2 \to \mathbf{R}^1$ case, in which f means $f(x, y)$, $\mathbf{p}$ means $[a, b]$ and $\mathbf{h}$ means $[x-a, y-b]$?
(b) What are the components of matrix $f'(\mathbf{p})$ in the $\mathbf{R}^2 \to \mathbf{R}^2$ case, in which $f(x, y)$ is itself a point of $\mathbf{R}^2$?

292. *Spherical coordinates I.* Points on the unit sphere $x^2 + y^2 + z^2 = 1$ can be described parametrically by

$$\begin{aligned} x &= \sin\phi\cos\theta \\ y &= \sin\phi\sin\theta \\ z &= \cos\phi, \end{aligned}$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. The angle θ is the angle usually called *longitude*, and the angle ϕ is the *complement* of the angle usually called *latitude*. This defines a mapping $f : \mathbf{R}^2 \to \mathbf{R}^3$, which "wraps" the rectangle $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ around the sphere. The $\phi\theta$ grid is transformed into the familiar latitude-longitude grid on the sphere.
(a) Write the 3×2 matrix $f'(\phi, \theta)$.
(b) Explain why the columns of $f'(\phi, \theta)$ are vectors tangent to the sphere at $f(\phi, \theta)$.
(c) Calculate $\mathbf{n}(\phi, \theta)$, the cross product of these column vectors.
(d) Show that the length of $\mathbf{n}(\phi, \theta)$ is $\sin\phi$. Use this *Jacobian* in a double integral to confirm that the surface area of the unit sphere is indeed 4π.

293. Calculate the area of the unit sphere that is found between the parallel planes $z = a$ and $z = b$, where $-1 \le a \le b \le 1$. You should find that your answer depends only on the separation between the planes, not on the planes themselves.

294. The centroid of the unit sphere is the origin, and the centroid of the unit hemisphere defined by $0 \le z$ is $(0, 0, \frac{1}{2})$. Find the centroid of that part of the unit sphere that is found in the first octant. Because of symmetry, you need only find one coordinate.

295. *Spherical coordinates II.* By using spheres of varying radius in addition to the angles ϕ and θ, every point of xyz-space can be given new coordinates. The coordinate map

$$f(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

is obtained by simply inserting ρ into each of the equations in item 1 above. The symbol ρ (the Greek r) stands for $\sqrt{x^2 + y^2 + z^2}$, the distance to the origin. The infinite prism $0 \le \rho$ and $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ is mapped by f onto all of xyz-space.
(a) Make calculations that justify the Jacobian formula $dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.
(b) Make a sketch of a small "spherical brick" whose volume is $\rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$.

296. Interpret the arclength differential

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

in terms of local multipliers.

297. Sketch the first-quadrant region $\mathcal{R}$ defined by $225 \le 9x^2 + 25y^2 \le 900$. Integrate the function $f(x,y) = xy$ over $\mathcal{R}$. (*Hint*: Consider the quasi-polar change of variables $(x,y) = (5u\cos t, 3u\sin t)$.)

298. A surface parametrization $(x(t,u), y(t,u), z(t,u))$, which expresses x, y, and z in terms of variables t and u, is a mapping of type $\mathbf{R}^2 \to \mathbf{R}^3$. Assuming that the coordinate functions are all differentiable, the Jacobian of this transformation can be calculated by the formula

$$\sqrt{\left|\frac{\partial(y,z)}{\partial(t,u)}\right|^2 + \left|\frac{\partial(z,x)}{\partial(t,u)}\right|^2 + \left|\frac{\partial(x,y)}{\partial(t,u)}\right|^2}$$

where each pair of absolute value signs indicate the determinant of a 2×2 matrix. Explain this formula.

299. Describe the location of the centroid of a homogeneous hemispherical solid.

300. Calculate the Jacobian multiplier for the linear mapping $(x,y,z) = (t, u, at + bu + c)$.

301. Consider the region $\mathcal{E}$ enclosed by the ellipse $x^2 - 2xy + 2y^2 = 25$. Show that $\mathcal{E}$ is the image of a simpler region in the tu-plane, by means of the linear coordinate change $(x,y) = (3t + u, t + 2u)$. Find the area of $\mathcal{E}$.

302. In the Cartesian coordinate system for $\mathbf{R}^3$, equations such as $x = 3$ and $z = -2$ represent planes. In the spherical coordinate system, what configurations are represented by the equations $\rho = 5$, $\phi = 0.7$, and $\theta = 4.2$?

303. What portion of the volume of the unit sphere is contained in the cone $\phi \le \beta$? Your answer will of course depend on β, and should have a predictable value when $\beta = \frac{1}{2}\pi$.

304. Describe the location of the centroid of the solid spherical sector described in the preceding question. Your answer will of course depend on β, and should agree with the answer you found for item 3 above.

305. Given a 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, its *determinant* is, except for sign, the volume of the parallelepiped defined by the three column vectors $\mathbf{u} = [a,d,g]$, $\mathbf{v} = [b,e,h]$, and $\mathbf{w} = [c,f,i]$. Show that it can be evaluated as a *triple scalar product* in any of the equivalent forms $\mathbf{u}\bullet(\mathbf{v} \times \mathbf{w})$, $\mathbf{v}\bullet(\mathbf{w} \times \mathbf{u})$, or $\mathbf{w}\bullet(\mathbf{u} \times \mathbf{v})$, all of which equal $aei + bfg + cdh - afh - bdi - ceg$. The determinant is positive if and only if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a right-handed coordinate system.

306. Use the preceding to obtain the spherical-coordinate Jacobian $\rho^2 \sin\phi$.

307. Given three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbf{R}^3$, they determine a *tetrahedron* (a triangular pyramid). Express the volume of this tetrahedron in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

308. In terms of the spherical coordinates ρ, ϕ, and θ, describe **(a)** the xy-plane; **(b)** the plane $z = 2$; **(c)** the cylinder $x^2 + y^2 = 3$; **(d)** the plane $x = 4$.

309. The cylinder $x^2 + y^2 = 25$ is cut by the plane $2x + 6y + 3z = 42$. The intersection curve is an ellipse. Find its area.

310. Let $f(u, v) = \left(\sqrt{2u}\cos v, \sqrt{2u}\sin v\right)$.
(a) Calculate the matrix $f'(u, v)$.
(b) Show that f is an *area-preserving* transformation of coordinates.

311. Consider the surface $\mathcal{D}$ in $\mathbf{R}^3$ defined for $0 \le t \le 2\pi$ and $0 \le u \le 2\pi$ by

$$\begin{aligned} x &= (5 + 2\cos u)\cos t \\ y &= (5 + 2\cos u)\sin t \\ z &= 2\sin u \end{aligned}$$

(a) Confirm that the point $P = \left(\frac{124}{25}, \frac{93}{25}, \frac{8}{5}\right)$ is on this surface, by finding the corresponding values of t and u.
(b) Find components for two nonzero, nonparallel vectors that are tangent to $\mathcal{D}$ at P.
(c) Find the cross product of the vectors you found in (b).
(d) Find the surface area of $\mathcal{D}$.

312. Consider the region $\mathcal{S}$ in $\mathbf{R}^3$ that is enclosed by the cone $\phi \le \kappa$ and the concentric spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $a < b$. Find the volume of $\mathcal{S}$, expressed in terms of a, b, and κ.

313. (Continuation) Find the z-coordinate of the centroid of the region $\mathcal{S}$. Your answer will of course depend on the values of a, b, and κ.

314. (Continuation) Find the limiting position of the z-coordinate as $a \to b$.

315. Let $\mathcal{R}$ be the trapezoidal region of $\mathbf{R}^2$ whose vertices are $(1, 0)$, $(3, 0)$, $(0, 3)$, and $(0, 1)$. Integrate $f(x, y) = e^{\frac{x-y}{x+y}}$ over $\mathcal{R}$.

316. Use spherical coordinates to find the average distance from $(0, 0, 1)$ to an arbitrary point on the unit sphere $x^2 + y^2 + z^2 = 1$.

317. Let $P = (0, 0, a)$, where $1 < a$, and let Q be a point of the unit ball $x^2 + y^2 + z^2 \le 1$. It should be clear that $a - 1 \le PQ \le a + 1$. Calculate the average value of $1/PQ$, which therefore should be somewhere between $\dfrac{1}{a+1}$ and $\dfrac{1}{a-1}$. It is an interesting number.

The Newtonian theory of gravitation allows one to work with *point masses*. In other words, given a homogeneous mass, the gravitational force it exerts on a remote object can be calculated (using the inverse-square formula) by pretending that all the mass is concentrated at the centroid of the homogeneous mass. The justification for the special case of a homogenous spherical mass is item 2 below. The corresponding theorem for gravitational potential is item 1 above.

Let $P = (0, 0, a)$, where $1 < a$, and let Q be a point of the unit ball $x^2 + y^2 + z^2 \le 1$. The force exerted by a small amount of mass m_1 at Q on a mass m_2 at P is directed toward Q and its magnitude is Gm_2m_1/PQ^2, where G is a gravitational constant. Because of the spherical symmetry, the net result of all such forces acting on m_2 will be directed toward the center of the sphere (the other components of the force cancel out). To obtain the only component of interest, simply multiply the force magnitude by $\cos\alpha$, where α is the angle formed by the vectors that point from P to Q and from P to the sphere center. Thus the resulting magnitude can be calculated as

$$\int_0^{2\pi}\int_0^1\int_0^\pi \frac{Gm_2\cos\alpha}{u^2}\cdot\frac{M\rho^2\sin\phi\,d\phi\,d\rho\,d\theta}{4\pi/3}$$

where M is the total mass of the spherical object (whose volume is of course $4\pi/3$), and $u^2 = PQ^2 = a^2 + \rho^2 - 2a\rho\cos\phi$.

318. Show that the value of the force integral is equal to GMm_2a^{-2}. To begin, multiply numerator and denominator of the integrand by u, and notice that $u\cos\alpha$ can be replaced by $a - \rho\cos\phi$. Integration by parts works very well here.

The simplicity of the result is remarkable, especially when one realizes that other inviting possiblities do not work out so neatly.

319. The average value of PQ is not a. Verify that it is $a + \dfrac{1}{5a}$.

320. The average value of $\dfrac{1}{PQ^2}$ is not $\dfrac{1}{a^2}$. Verify that it is $\dfrac{3}{2} - \dfrac{3}{4a}(a^2-1)\ln\dfrac{a+1}{a-1}$.

321. At first glance, the preceding answer does not look right! With the help of the Maclaurin identity

$$\ln\frac{a+1}{a-1} = \ln\frac{1+(1/a)}{1-(1/a)} = 2\left(\frac{1}{a}\right) + \frac{2}{3}\left(\frac{1}{a}\right)^3 + \frac{2}{5}\left(\frac{1}{a}\right)^5 + \cdots$$

show that the average value can be rewritten in the reassuring form

$$\frac{1}{a^2} + \frac{3}{15}\left(\frac{1}{a}\right)^4 + \frac{3}{35}\left(\frac{1}{a}\right)^6 + \cdots = 3\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\left(\frac{1}{a}\right)^{2n}.$$

322. *Cylindrical coordinates* are a self-explanatory extension of polar coordinates to 3-dimensional space. The coordinate transformation is $(x, y, z) = (r\cos\theta, r\sin\theta, z)$, where $r^2 = x^2 + y^2$. Notice the distinction between the polar variable r and the spherical variable ρ. Use a determinant to justify the equation $dx\,dy\,dz = r\,dr\,d\theta\,dz$.

323. Let $\mathcal{P}$ be the region in $\mathbf{R}^3$ defined by $0 \le z \le 4 - x^2 - y^2$. Use cylindrical coordinates to find the volume of $\mathcal{P}$. Then find the z-coordinate of the centroid of $\mathcal{P}$.

324. Let $F(x, y) = \left(\sqrt{x^2 + y^2}\,, \arctan\dfrac{y}{x}\right)$, restricting your attention to the right half-plane $\mathcal{H}$ where $0 < x$. Let $G(r, \theta) = (r\cos\theta, r\sin\theta)$, restricting attention to the semi-infinite strip $\mathcal{S}$ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ and $0 < r$.
(a) What is the relationship between the functions F and G?
(b) Calculate the 2×2 matrices F' and G'. How are these matrices related?
(c) Calculate the determinant of F'. Could you have predicted the result?

325. The *unit disk* in $\mathbf{R}^n$ is defined to consist of all points whose distance from the origin is at most 1. It is denoted $\mathbf{D}^n$. Let d_n be the *content* of $\mathbf{D}^n$. Thus $d_1 = 2$, $d_2 = \pi$, and $d_3 = \frac{4}{3}\pi$.
(a) Explain why $d_4 = \displaystyle\int_{-1}^{1} d_3\left(\sqrt{1 - x^2}\right)^3 dx$.
(b) With the help of the substitution $x = \cos t$, show that $d_4 = \frac{1}{2}\pi^2$.
(c) In general, show that $d_n = d_{n-1}\displaystyle\int_0^{\pi} (\sin t)^n\,dt$
(d) Let I_n be the value of the integral $\displaystyle\int_0^{\pi} (\sin t)^n dt$. It is evident that $I_0 = \pi$ and that $I_1 = 2$. For $2 \le n$, use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}$.
(e) Deduce that $I_{2m} = \pi\left(\dfrac{1}{2}\right)^{2m}\dbinom{2m}{m}$ and that $I_{2m+1} = \dfrac{2^{2m+1}}{2m+1}\dbinom{2m}{m}^{-1}$.
(f) Notice that $I_{2m}I_{2m+1} = \dfrac{2\pi}{2m+1}$ and that $I_{2m-1}I_{2m} = \dfrac{2\pi}{2m}$. Use these to help you verify the formulas $d_{2m} = \dfrac{\pi^m}{m!}$ and $d_{2m+1} = \dfrac{2^{2m+1}m!\pi^m}{(2m+1)!}$.

326. Derive the corresponding formulas for the content s_n of the n-sphere $\mathbf{S}^n$. ($s_2 = 4\pi$)

327. Find the range of values of $f(x,y) = (x^2+y^2)^{-1}$ on the curve $9x^4 + 16y^4 = 3600$.

328. Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2}\int_1^2 e^x \sin\frac{y}{x}\,dx\,dy$.

329. It is customary to describe a curve in $\mathbf{R}^3$ by expressing its Cartesian coordinates x, y, and z as functions of t. As you know, the length of a differentiable arc is the value of

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

It is also possible to describe a curve in $\mathbf{R}^3$ by expressing its spherical coordinates ρ, ϕ, and θ as functions of t. Show that the length of a differentiable arc is

$$\int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \left(\rho\sin\phi\frac{d\theta}{dt}\right)^2 + \left(\rho\frac{d\phi}{dt}\right)^2}\,dt.$$

One way to proceed is to use the Product Rule and a lot of algebra, but it is also possible to think geometrically and save a lot of work.

330. The cone $\sin\phi = \frac{5}{13}$ is sliced by the plane $4x + 4y + 7z = 112$. One of the intersection points is $P = (3,4,12)$. The intersection curve is an ellipse. If this cone were cut along the ray from the origin through P, it could be "unrolled", thus forming an infinite sector.
(a) What is the angular size of this sector?
(b) The ellipse has one point for each value of the longitude θ, for $0 \le \theta < 2\pi$, so it should be possible to express all the other variables in terms of θ. Try it.
(c) What is the range of ρ-values on the ellipse?
(d) When the cone is "unrolled", the ellipse becomes a curve that connects two points on the radial edges of the sector. Explain why this curve cannot be a (straight) segment.

331. Let $N = (0,0,1)$ and Q be an arbitrary point of the unit disk $\mathbf{D}^3$. Find the average of all lengths NQ, as Q ranges over $\mathbf{D}^3$.

332. Show that the function $p(t) = \dfrac{1}{\sqrt{2\pi}}\,e^{-t^2/2}$ has the following two properties:
(a) $1 = \displaystyle\int_{-\infty}^{\infty} p(t)\,dt$
(b) $1 = \displaystyle\int_{-\infty}^{\infty} t^2 p(t)\,dt$
This is the so-called *standard normal distribution of probability.*

333. Consider the cylinder defined by $r = 1$ and $0 \le z \le 2$ in cylindrical coordinates. Here is an attempt to confirm the familiar value 4π for the lateral surface area of this cylinder, by inscribing lots of little polygons (triangles):
(a) Divide the height into m equal parts, using the values $z = 0$, $z = 2/m$, $z = 4/m$, $z = 6/m$, etc. At each of these heights, draw the circle on the cylinder that is parallel to the base circle.
(b) Divide the base circle into n equal parts, using the points $\theta = 0$, $\theta = 2\pi/n$, $\theta = 4\pi/n$, $\theta = 6\pi/n$, etc. Divide the circle at $z = 2/m$ into n parts as well, but offset the points a half-step by using $\theta = \pi/n$, $\theta = 2\pi/n + \pi/n$, etc. Use these $2n$ points to inscribe $2n$ congruent isosceles triangles in the thin (height $2/m$) cylinder between the two circles; half of the triangles point upward, and half point downward. Find the area of one of the triangles.
(c) Divide the circle at $z = 4/m$ into n parts, using the same values of θ as for the base circle, then join each of these n new points to the nearest two points on the circle at $z = 2/m$, thus creating $2n$ new triangles that are congruent to those found in step (b). Continuing in this way, you will inscribe $2mn$ congruent triangles in the cylinder.
(d) Let $m = n^2$, and write a formula for the combined area of the $2n^3$ inscribed triangles. Then let n approach infinity. It is of course expected that the sum of the triangle areas will approach the lateral surface area of the cylinder. To evaluate this limit, it will be convenient to make use of the two familiar results

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}\,.$$

Is the answer what you expected?

334. Express the arc length integral $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ in terms of the cylindrical variables r, θ, and z.

335. *Stereographic projection.* The transformation

$$x = \frac{2t}{t^2+u^2+1} \quad , \quad y = \frac{2u}{t^2+u^2+1} \quad , \quad z = \frac{t^2+u^2-1}{t^2+u^2+1}$$

maps $\mathbf{R}^2$ onto a surface $\mathcal{S}$ in $\mathbf{R}^3$.
(a) Show that circles centered at the origin in $\mathbf{R}^2$ are mapped to circles on $\mathcal{S}$.
(b) One of the points on $\mathcal{S}$ is $P = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, which is the image of $t = 2$ and $u = 1$. Find two vectors that are tangent to $\mathcal{S}$ at P.

336. Consider that part of the unit sphere $\rho = 1$ that lies inside the circular cylinder $r = \cos\theta$ and above the plane $z = 0$. Find the area of this surface.

337. Consider the interesting function

$$F(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}\,, & \text{if } 0<x^2+y^2\,;\\ 0, & \text{otherwise.}\end{cases}$$

(a) Show that $f_x(0,0)$ and $f_y(0,0)$ are both zero.
(b) Calculate $f_{\mathbf{u}}(0,0)$ for a general unit vector $\mathbf{u}=a\mathbf{i}+b\mathbf{j}$.
(c) Conclude that f is not differentiable at $(0,0)$.
(d) Show that f is discontinuous at $(0,0)$.

338. Consider that part of the unit 3-ball $\rho\le 1$ that lies inside the circular cylinder $r=\cos\theta$ and above the plane $z=0$. Find the volume of this region.

339. Finding the point on the sphere $x^2+y^2+z^2=225$ that is farthest from $P=(9,12,20)$ can be done by simple geometry. Do so. Then describe the problem in terms of maximizing a function $f(x,y,z)$ subject to a constraint $g(x,y,z)=225$. Give a detailed description of how the level surfaces of f intersect the constraining surface $\mathcal{S}$, thus producing level curves of f on $\mathcal{S}$.

340. Let $\mathcal{P}$ be the paraboloid $3z=x^2+y^2$, and $f(x,y,z)=(x-1)^2+(y-0)^2+(z-4)^2$. When (x,y,z) is constrained to lie on $\mathcal{P}$, f has two local extrema and a saddle point. Find coordinates for these three points, and describe the configuration of curves on $\mathcal{P}$ that result from intersecting $\mathcal{P}$ with level surfaces $f(x,y,z)=k$.

341. If gazillions of points were randomly selected from the unit 3-ball $\rho\le 1$, what would the average of their ρ-values be?

342. The semicircular sector defined by $x^2+y^2\le 36$ and $0\le y$ can be rolled up to form a cone, which can be placed so that its vertex is at the origin and its axis of symmetry is the positive z-axis. In the process, the segment that joins $(0,6)$ to $(6,0)$ is transformed into a curve $\mathcal{C}$ on the cone.
(a) Describe the cone using spherical coordinates.
(b) Is $\mathcal{C}$ a planar curve? Give your reasons.

343. Carefully interpreted, it is true that $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$. Explain.

344. Carefully interpreted, it is true that $\dfrac{\partial(x,y)}{\partial(u,v)}\cdot\dfrac{\partial(u,v)}{\partial(x,y)}=1$. Explain.

345. Give an example that shows that $\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial u}=1$ is *not* true.

346. Evaluate $\displaystyle\int_0^3 \int_{y^2}^9 ye^{-x^2}\,dx\,dy$.

347. The region $\mathcal{R}$ is bounded by the planes $z = 0$ and $z = 2y$ and by the parabolic cylinder $x^2 + y = 4$. Find the z-coordinate of the centroid of $\mathcal{R}$.

348. Restrict your attention to points on the sphere $x^2 + y^2 + z^2 = 2$ in the first octant (where x, y, and z are all positive).
(a) Find the maximum value of $f(x, y, z) = x^4y^9z^{36}$.
(b) Find spherical coordinates ρ, ϕ, and θ for the point where f attains its maximal value.

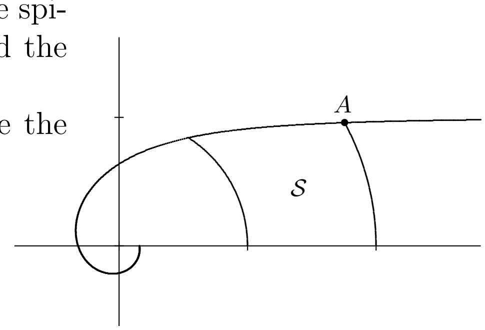
349. The *heat equation* (also known as the *diffusion equation*) is a partial differential equation that describes how temperatures vary with respect to position and with respect to time. In two dimensions, it is

$$k\,\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\,,$$

where $T = T(x, y, t)$ is the temperature at position (x, y) and at time t, and where k is a constant that depends on the physical attributes of the material (thermal conductivity, for example). Of special interest is the *steady-state* equation $0 = T_{xx} + T_{yy}$ (also known as *Laplace's equation*), obtained by setting the time derivative to zero.
(a) If $T(x, y)$ is a steady-state temperature distribution, then every point on the graph of the equation $z = T(x, y)$ is a saddle point. Explain this remark.
(b) If m is a constant, then any function of the form $T(x, y) = e^{-mx}\sin my$ is a solution to the steady-state equation. Verify that this is true.
(c) Let $T(x, y) = \displaystyle\sum_{n=1}^{N} c_n e^{-nx}\sin ny$ where each c_n is constant. Verify that T describes a steady-state temperature distribution.

350. Find the area of the surface $z = xy$ that is contained inside the cylinder $x^2 + y^2 = 8$.

351. The diagram shows the region $\mathcal{S}$ bounded by the spiral $r\theta = 1$, the circle $r = 1$, the positive x-axis, and the circle $r = 2$.
(a) Find coordinates x and y for the point A where the spiral intersects the circle $r = 2$.
(b) Find the area of $\mathcal{S}$.



352. Consider the transformations $\mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$F(u,v) = (u + u^2 - 2uv + v^2\,,\ v + u^2 - 2uv + v^2\,)$$
$$G(x,y) = (x - x^2 + 2xy - y^2\,,\ y - x^2 + 2xy - y^2\,)$$

Show that each transformation is
(a) area-preserving;
(b) *one-to-one* — this means that different points have different images;
(c) *onto* $\mathbf{R}^2$ — this means that the range is all of $\mathbf{R}^2$.

353. (Continuation) Calculate both compositions $F \circ G$ and $G \circ F$.

354. Show that the linear transformation $(x,y) = \left(\dfrac{u + v\sqrt{3}}{2}, \dfrac{v - u\sqrt{3}}{2}\right)$ is area-preserving.

355. What condition on the coefficients a, b, c, k, m, and n ensures that the generic linear transformation $(x,y) = (a + bu + cv\,,\ k + mu + nv)$ is area-preserving?

356. As you have seen, the nonlinear transformation $(x,y) = (u^2 - v^2, 2uv)$ distorts regions and alters their areas. It does have a special property, however; at every point except the origin, this transformation is *conformal*, which means that it preserves the sizes of angles.
(a) Find an example to illustrate this statement.
(b) Prove the general assertion.

357. Consider the linear transformation $(x,y) = (3u + v, u + 2v)$, which transforms the circle $u^2 + v^2 = 2u + 2v$ into the ellipse $x^2 - 2xy + 2y^2 - 2x - 4y = 0$. The circle circumscribes the square whose vertices are $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$; the ellipse circumscribes the parallelogram whose vertices are $(0,0)$, $(6,2)$, $(8,6)$, and $(2,4)$.
(a) Is $(8,6)$ a vertex of the ellipse?
(b) The image of the center of the circle is $(4,3)$. Is this the center of the ellipse?

358. Consider the surface $\mathcal{S}$ in $\mathbf{R}^3$ defined by revolving an arc $y = f(x)$ for $a \le x \le b$ around the x-axis. Assume that f is a differentiable function and that f' is continuous.
(a) Show that $\mathcal{S}$ can be parametrized by $(x,y,z) = (t, f(t)\cos u, f(t)\sin u)$.
(b) Calculate the Jacobian for this parametrization.
(c) Set up an integral with respect to t and u for the area of $\mathcal{S}$.

359. Show that any transformation of the form $(x,y) = (u + f(u - v), v + f(u - v))$ is area-preserving, assuming that f is a differentiable real-valued function of a real variable.

360. Consider the figure-eight curve obtained by intersecting the unit sphere $\rho = 1$ and the cylinder $r = \cos\theta$. Find a parametrization for the entire curve.

361. *More fun with iterated integrals.* Evaluate the following:
(a) $\displaystyle\int_0^1 \int_0^1 2xye^{xy^2}\,dxdy$ **(b)** $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\,dxdy$ **(c)** $\displaystyle\int_0^1 \frac{x^5 - x^3}{\ln x}\,dx$

362. For each nonnegative integer n, let $a_n = \displaystyle\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{2\sqrt{x}}\,dx.$
(a) Show that $|a_n|$ decreases monotonically to 0.
(b) Conclude that the series $\sum_{n=0}^{\infty} a_n$ is convergent. Using advanced methods, it can be shown that the sum is $\sqrt{\pi/8}$.

363. (Continuation) Evaluate the improper integral $\displaystyle\int_0^{\infty} \sin\left(t^2\right)\,dt.$

364. (Continuation) Show that $\dfrac{\pi}{4} = \displaystyle\lim_{m\to\infty} \int_0^m \int_0^m \sin\left(x^2 + y^2\right)\,dxdy\,.$

365. Show that $\displaystyle\lim_{m\to\infty} \int_0^{\pi/2} \int_0^m \sin\left(r^2\right)\,r\,drd\theta$ does not exist.

366. The preceding two questions show that it is difficult to define convergence of improper multivariable integrals when the integrand is not of constant sign. Explain.

367. If the improper integral $\displaystyle\int_1^{\infty} \int_1^{\infty} \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\,dxdy$ has a meaningful value, it is clear that the value *should* be 0. When this expression is evaluated as an iterated integral, however, a nonzero value is obtained! Verify.

368. Let $\mathcal{R}$ be the region $(x-1)^2 + y^2 \le 1$. Evaluate $\displaystyle\int\int_{\mathcal{R}} \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\,dxdy\,.$

369. The integral $\displaystyle\int_0^1 \int_0^1 \frac{1}{1 - xy}\,dxdy$ is improper but convergent. With the help of a geometric series, show that the value of the integral is $\sum_{n=1}^{\infty} n^{-2}$.

370. Given a rectangle $\mathcal{R}$, let $\partial\mathcal{R}$ be the boundary of $\mathcal{R}$, considered as a piecewise differentiable path, parametrized in a counter-clockwise direction. Let P and Q be functions that are differentiable on $\mathcal{R}$. Show that the line integral $\int_{\partial\mathcal{R}} P\,dx + Q\,dy$ has the same value as the integral $\displaystyle\int\int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$. This is an example of *Green's Theorem*.

371. (Continuation) Show the result remains true if $\mathcal{R}$ is any *type I* region, which means that $\mathcal{R}$ is defined by inequalities $a \le x \le b$ and $f(x) \le y \le g(x)$, where f and g have continuous derivatives. *Hint*: Consider the special cases $P = 0$ and $Q = 0$ first.

372. State and prove Green's Theorem for type II regions.

373. When $P(x,y) = -\frac{1}{2}y$ and $Q(x,y) = \frac{1}{2}x$, Green's Theorem is interesting. Explain.

374. Let $\mathcal{S}$ be the region consisting of those points (x,y) for which $1 \le x^2 + y^2 \le 9$ is true, but for which $0 < x$ and $y < 0$ are not both true. Is region $\mathcal{S}$ covered by our treatment of Green's Theorem? Does the conclusion of the theorem hold for the example defined by $P(x,y) = 2xy$ and $Q(x,y) = 2x + x^2$?

375. (Continuation) Using the familiar polar-coordinate mapping, show that $\mathcal{S}$ is the image of a rectangular region $\mathcal{R}$ in $r\theta$-space.

376. Without evaluating them, show that $\int\int_{x^2+y^2\le 4}(2x\cos(x^2+y^2) + 2y\sin(x^2+y^2))\,dx\,dy$ and $\int_{x^2+y^2=4}\cos(x^2+y^2)\,dx + \sin(x^2+y^2)\,dy$ have the same value.

The next question is made more difficult by an unfortunate notational practice: The letter P is used to name a function of x and y, but then P is also used to name the function that results from P by replacing all occurrences of x and y by chosen functions of r and θ.

377. Applying a change of variables $x = x(r,\theta)$ and $y = y(r,\theta)$ to an $r\theta$-rectangle $\mathcal{R}$ produces a region $\mathcal{S}$ in xy-space. This allows you to perform a substitution on each of the two integrals that appear in the statement of Green's Theorem for $\mathcal{S}$.

(a) Explain why $\displaystyle\int\int_{\mathcal{S}}(Q_x - P_y)\,dx\,dy = \int\int_{\mathcal{R}}(Q_x - P_y)(x_r y_\theta - x_\theta y_r)\,dr\,d\theta$.

(b) Explain why $\displaystyle\int_{\partial\mathcal{S}} P\,dx + Q\,dy = \int_{\partial\mathcal{R}} (P\cdot x_r + Q\cdot y_r)\,dr + (P\cdot x_\theta + Q\cdot y_\theta)\,d\theta$. It will be necessary to apply the Chain Rule to dx/dt and to dy/dt.

(c) Calculate $\dfrac{\partial}{\partial r}(P\cdot x_\theta + Q\cdot y_\theta)$, thinking of $P(x(r,\theta), y(r,\theta))$ and $Q(x(r,\theta), y(r,\theta))$ as functions of r and θ.

(d) In a similar fashion, calculate $\dfrac{\partial}{\partial\theta}(P\cdot x_r + Q\cdot y_r)$.

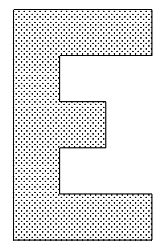
(e) By comparing your answers to the previous four parts, conclude that Green's Theorem holds for the transformed region $\mathcal{S}$.

378. Given continuous functions P and Q, and a continuously differentiable path $\mathcal{C}$, the line integral $\int_{\mathcal{C}} P\,dx + Q\,dy$ is defined. Notice that the integrand is a dot product:

$$P\,dx + Q\,dy = \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt = [P,Q]\bullet\left[\frac{dx}{dt}\,,\,\frac{dy}{dt}\right]dt$$

This construction has many applications, especially in physics. When $\mathcal{C}$ is a closed path, this integral is sometimes called the *circulation* of the vector field $[P,Q]$ around $\mathcal{C}$. A positive value indicates a tendency for the vector field to point in the same direction as the motion along the path. Explain.

379. Green's Theorem is valid for the region $\mathcal{E}$ shown at right. Explain why. *Hint*: Notice that $\mathcal{E}$ can be expressed as a union of rectangles, any two of which have at most a single segment in common.



380. Find the circulation of the vector field $\left[0\,,\,\dfrac{1}{1+x^2}\right]$ around the piecewise linear path that goes from $(2,0)$ to $(2,1)$ to $(1,1)$ to $(1,0)$ to $(2,0)$.

381. Suppose that $Q_x = P_y$ throughout a region $\mathcal{R}$. What deductions can you make?

382. Given a parametrized surface $\mathcal{S}$ in $\mathbf{R}^3$, and three real functions M, N, and P defined on $\mathcal{S}$, the expression $\int_{\mathcal{S}} M\,dy\,dz + N\,dz\,dx + P\,dx\,dy$ is called a *surface integral*. The surface parametrization $(x(t,u), y(t,u), z(t,u))$ maps some tu-rectangle $\mathcal{R}$ onto $\mathcal{S}$, and the integral is thus evaluated

$$\int\!\!\int_{\mathcal{R}}\left(M\frac{\partial(y,z)}{\partial(t,u)} + N\frac{\partial(z,x)}{\partial(t,u)} + P\frac{\partial(x,y)}{\partial(t,u)}\right)dt\,du\,,$$

where $M = M(x(t,u), y(t,u), z(t,u))$ is viewed as a function of t and u, and $\partial(y,z)/\partial(t,u)$ is the determinant of a 2×2 array of partial derivatives.

For example, let $\mathcal{S}$ be the upper half of the unit sphere — those points in $\mathbf{R}^3$ that satisfy $x^2+y^2+z^2=1$ and $0\le z$. Calculate

(a) $\int_{\mathcal{S}}\ x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ **(b)** $\int_{\mathcal{S}}\ x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$

383. (Continuation) In general, the integrand is the projection of $\mathbf{v} = [M,N,P]$ onto a unit vector $\mathbf{n}_{\mathcal{S}}$ that is normal to $\mathcal{S}$. Explain. In applications, $\int_{\mathcal{S}} M\,dy\,dz + N\,dz\,dx + P\,dx\,dy$ is often called the *flux* of $\mathbf{v}$ through the surface $\mathcal{S}$, and it can be written simply $\int_{\mathcal{S}} \mathbf{v}\bullet\mathbf{n}_{\mathcal{S}}$.

384. Let $\mathcal{A}$ be a plane region, whose area is a, and let a_1, a_2, and a_3 be the areas of the regions obtained by projecting $\mathcal{A}$ onto the planes $x=0$, $y=0$, and $z=0$, respectively. Prove that $a^2 = a_1^2 + a_2^2 + a_3^2$.

385. Calculate the flux of the vector field $[x,y,z]$ through the triangular surface whose vertices are $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, assuming that a, b, and c are all positive.

386. Given a cube $\mathcal{C}$, let $\partial\mathcal{C}$ be the boundary of $\mathcal{C}$, considered as a piecewise differentiable surface (there are six sections). It is assumed that the parametrizations for each section are chosen so that the normal vectors all point outwards. Let $[M, N, P]$ be a vector field that is defined continuously throughout $\mathcal{C}$. Show that the flux integral

$$\int_{\partial\mathcal{C}} M\,dy\,dz + N\,dz\,dx + P\,dx\,dy$$

has the same value as the integral

$$\int\int\int_{\mathcal{C}} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)\,dx\,dy\,dz\,.$$

This is an example of the *Divergence Theorem*, also known as *Gauss's Theorem.*

387. Let $\mathcal{R}$ be a region in $\mathbf{R}^3$, for which $\partial\mathcal{R}$ is a piecewise differentiable surface (which may have several sections). Let $\mathbf{v}$ be a continuously differentiable vector field defined throughout $\mathcal{R}$. The Divergence Theorem states that

$$\int_{\mathcal{R}} \nabla\bullet\mathbf{v} = \int_{\partial\mathcal{R}} \mathbf{v}$$

where ∇ is the familiar "differential operator" defined (for $\mathbf{R}^3$) by

$$\nabla = \left[\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right] = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\,,$$

and usually read "del." Use this theorem to help you calculate the flux of the field $[5x^2, 4y, 3]$ through the unit sphere $\mathcal{S}$, using an outward-pointing normal for $\mathcal{S}$.

388. Consider the vector fields $\mathbf{v} = [xz - xy, yx - yz, zy - zx]$ and $\mathbf{w} = [y + z, z + x, x + y]$, and let $\mathcal{T}$ be the triangular surface determined by the points $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$. Both of the following questions have the same answer.
(a) Calculate the circulation of $\mathbf{v}$ around the piecewise linear path $\partial\mathcal{T}$.
(b) Calculate the flux of $\mathbf{w}$ through the surface $\mathcal{T}$, which has normal vector $[1, 2, 3]$.

389. Given a differentiable vector field $\mathbf{v} = [P, Q, R]$ defined in $\mathbf{R}^3$, define

$$\mathbf{curl}(\mathbf{v}) = \nabla \times \mathbf{v} = \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\,,\,\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\,,\,\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right].$$

Let $\mathcal{S}$ be a surface and $\partial\mathcal{S}$ be its (oriented) boundary. *Stokes's Theorem* states that

$$\int_{\partial\mathcal{S}} \mathbf{v} = \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \bullet \mathbf{n}_{\mathcal{S}}$$

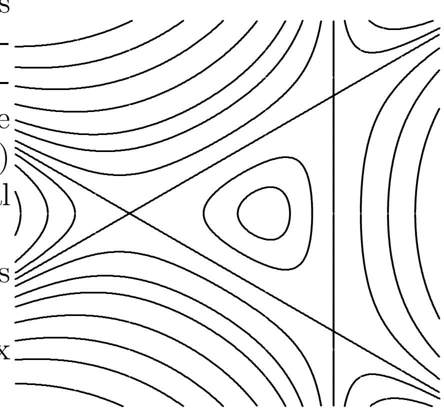
That is, the circulation of $\mathbf{v}$ around $\partial\mathcal{S}$ is equal to the flux of $\mathbf{curl}(\mathbf{v})$ through $\mathcal{S}$. Explain how Green's Theorem is a special case of this theorem.

390. For any field $\mathbf{v}$ or function f defined on $\mathbf{R}^3$, show that $\nabla\bullet(\nabla\times\mathbf{v}) = 0$ and $\nabla\times(\nabla f) = \mathbf{0}$.

391. Given a bounded region $\mathcal{R}$ in $\mathbf{R}^3$, whose boundary $\partial\mathcal{R}$ consists of surfaces that are differentiable, the volume of $\mathcal{R}$ is equal to the flux of the field $[\frac{1}{3}x, \frac{1}{3}y, \frac{1}{3}z]$ across $\partial\mathcal{R}$, assuming that each boundary component has an outward-pointing normal. Explain, and provide an illustration.

392. (Continuation) Find another field that has the same property as does $[\frac{1}{3}x, \frac{1}{3}y, \frac{1}{3}z]$.

393. Consider the vector field $\mathbf{v}$, defined throughout $\mathbf{R}^2$ by $\mathbf{v} = [2xy - y, x^2 + x - y^2]$. The diagram at right shows the *streamlines* of this field, rather than the vectors themselves. (In effect, the field vectors are interpreted as instantaneous velocity vectors, and the streamlines represent the solution trajectories for the resulting differential equation.)



(a) The diagram shows that the field $\mathbf{v}$ has four critical points, where $\mathbf{v} = \mathbf{0}$. Find these points.
(b) Show that this field has zero divergence (which means that it has no *sources* or *sinks*).
(c) Using an outward-pointing normal, calculate the flux of $\mathbf{v}$ across the circle $x^2 + y^2 = 2$.
(d) Put arrows on the trajectories, to show the directions of the field vectors.
(e) Calculate the circulation of $\mathbf{v}$ around the positively directed circle $x^2 + y^2 = 2$.

394. *The Fundamental Theorem of Calculus.* You have recently read several theorems that all take the form

$$\int_{\mathcal{C}} d\omega = \int_{\partial\mathcal{C}} \omega$$

in which the domain of integration (either $\mathcal{C}$ or $\partial\mathcal{C}$) is abstractly referred to as a *chain*, and the integrand (either $d\omega$ or ω) is called a *differential form.* In all cases, the degree of the form matches the dimension of the chain. For examples: a 1-form such as $P\,dx + Q\,dy + R\,dz$ must be integrated over a 1-chain that is a differentable path (or a collection thereof); and 2-forms such as $M\,dy\,dz + N\,dz\,dx + P\,dx\,dy$ or $(Q_x - P_y)dx\,dy$ must be integrated over a 2-chain that is a differentiable surface (or a collection thereof). The two integrals in the statement are linked by two linear operators: ∂ builds a chain from $\mathcal{C}$ by extracting the boundary of each piece of $\mathcal{C}$ (which lowers the chain dimension by 1); d converts a k-form into a $(k+1)$-form, as specified by the Chain Rule ($dM = M_x dx + M_y dy + \cdots$) and *skew-symmetry* (that means $dy\,dx = -dx\,dy$; notice that it also implies $dxdx = 0$).
For example, consider Green's Theorem, and let $\omega = P\,dx + Q\,dy$. Here is how d works:

$$\begin{aligned} d\omega &= d(P\,dx + Q\,dy) = (P_x\,dx + P_y\,dy)\,dx + (Q_x\,dx + Q_y\,dy)\,dy \\ &= P_x\,dx\,dx + P_y\,dy\,dx + Q_x\,dx\,dy + Q_y\,dy\,dy = (Q_x - P_y)\,dx\,dy \end{aligned}$$

Write similar explanations for the Divergence Theorem and Stokes's Theorem.

395. In its classic form, the Fundamental Theorem of Calculus deals with a 0-form $\omega = f$ (a function) and a 1-chain $\mathcal{C}$, which is a (directed) interval. Provide the remaining details.

A chain that arises from applying the operator ∂ is called a *boundary.* A chain $\mathcal{C}$ for which $\partial\mathcal{C} = 0$ is called a *cycle.* It is fundamental that all boundaries are cycles, which means $\partial(\partial\mathcal{C}) = 0$. This is a consequence of *orientation* — every element of a chain is stamped with an orientation, made possible by its parametrization, and these orientations must be carefully managed during the formation of a boundary.

For example, consider the solid cube $\mathcal{C}$ in $\mathbf{R}^3$; it becomes an oriented 3-chain by stamping it with the *right-hand rule* (familiar from the definition of $\mathbf{k} = \mathbf{i}\times\mathbf{j}$). The boundary 2-chain $\partial\mathcal{C}$ consists of six terms, each of which must also be stamped with an orientation (either *clockwise* or *counterclockwise*) that is derived from the cube orientation. The standard way of doing this is to stipulate that, for any face, the cube orientation matches the orientation that results from appending the outward-pointing normal vector to the designated face orientation; there is only one way to do this.

In the same way, the orientation on a 2-cell (a square face, for example) imposes an orientation (a *direction*) on each of the 1-cells that form its boundary.

It follows that the computation of $\partial(\partial\mathcal{C})$ leads one to consider $24 = 6 \cdot 4$ oriented segments (1-cells). It is intuitively clear that, if the preceding rules are followed exactly, each of the twelve edges of the cube will appear twice in this calculation, once with each of the two possible orientations. By convention, the two contributions of each edge are said to *cancel,* which is what $\partial(\partial\mathcal{C}) = 0$ means.

396. Consider the tetrahedral region $\mathcal{T}$ in $\mathbf{R}^3$, whose vertices are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. It is clear that $\partial\mathcal{T}$ has four terms. Suppose that they have been parametrized (and thus oriented) as follows, using nonnegative parameters t and u that satisfy $t + u \le 1$:

$$\begin{aligned}
\mathcal{S}_1 &: (x, y, z) = (t, u, 0)\\
\mathcal{S}_2 &: (x, y, z) = (0, t, u)\\
\mathcal{S}_3 &: (x, y, z) = (t, 0, u)\\
\mathcal{S}_4 &: (x, y, z) = (t, u, 1 - t - u)
\end{aligned}$$

Assuming that $\mathcal{T}$ carries the right-handed orientation, express $\partial\mathcal{T}$ as a sum of four terms, using ± 1 as coefficients to make $\partial(\partial\mathcal{C}) = 0$ true.

397. A differential form that arises from applying the operator d is called *exact.* A form ω for which $d\omega = 0$ is called *closed.* It is fundamental that all exact forms are closed, which means $d(d\omega) = 0$. This is a consequence of the sign-sensitivity that defines d, and of properties of derivatives. Verify $d(d\omega) = 0$ for the 1-form $\omega = P\,dx + Q\,dy + R\,dz$ in $\mathbf{R}^3$.

398. A natural question: if a 1-form is closed in a region $\mathcal{R}$, must it be exact? You have seen that the answer can be locally "yes", but globally "no." Explain, and provide an example. *Hint*: You are looking for a function with a prescribed gradient.

399. Consider the unit cube $\mathcal{C}$ in $\mathbf{R}^3$, defined by the simultaneous inequalities $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$, and let $\mathcal{S} = \partial\mathcal{C}$ be its boundary surface. For any point P on $\mathcal{S}$, let $f(P)$ be the distance — *measured along* $\mathcal{S}$ — from $O = (0,0,0)$ to P.
(a) Consider the corners $C_1 = (1,0,0)$, $C_2 = (1,0,1)$, $C_3 = (1,1,1)$, and $C_4 = (1,1,0)$. Show that $f(C_1) = 1$, $f(C_2) = \sqrt{2}$, $f(C_3) = \sqrt{5}$, and $f(C_4) = \sqrt{2}$.
(b) Show that the level curve $f(P) = 0.6$ is *piecewise circular* — composed of three quarter-circles that are joined continuously at their endpoints.
(c) Show that the level curve $f(P) = 1$ is also piecewise circular.
(d) Let $Q = (0.2, 0.5, 1.0)$ and $R = (0.5, 0.2, 1.0)$. Show that $f(Q) = 1.3 = f(R)$. Show that the level curve $f(P) = 1.3$ is piecewise circular. How many pieces are there?
(e) What is the range of values of f?
(f) Show that f, when restricted to the face where $z = 1$, is a piecewise differentiable function, by writing a recipe for calculating $f(x, y, 1)$.

400. Consider the rectangular box $\mathcal{B}$ in $\mathbf{R}^3$, defined by the three simultaneous inequalities $0 \le x \le 2$, $0 \le y \le 1$, and $0 \le z \le 1$, and let $\mathcal{S} = \partial\mathcal{B}$ be its boundary surface. For any point P on $\mathcal{S}$, let $f(P)$ be the distance — *measured along* $\mathcal{S}$ — from $O = (0,0,0)$ to P.
(a) Consider the corners $C_1 = (2,0,0)$, $C_2 = (2,0,1)$, $C_3 = (2,1,1)$, and $C_4 = (2,1,0)$. Show that $f(C_1) = 2$, $f(C_2) = \sqrt{5}$, $f(C_3) = \sqrt{8}$, and $f(C_4) = \sqrt{5}$.
(b) Let

$$Q = \left(\frac{48}{25}, 0, \frac{14}{25}\right),\ R = \left(\frac{6}{5}, \frac{3}{5}, 1\right),\ T = \left(\frac{-1+\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}, 1\right),\ U = \left(-1+\sqrt{3}, 1, 1\right)$$

Show that $f(Q) = f(R) = f(T) = f(U) = 2$.
(c) Show that the level curve $f(P) = 2$ is piecewise circular. How many pieces are there?
(d) Show that f, when restricted to the face where $z = 1$, is a piecewise differentiable function, by writing a recipe for calculating $f(x, y, 1)$.

401. (Continuation) Let $M = (2, 31/39, 21/26)$. Calculate $f(M)$. You will have to consider four piecewise linear paths from O to M. One of them is shorter than the other three.

402. (Continuation) Let $\mathcal{S}_2$ be the square face of $\partial\mathcal{B}$ on which $x = 2$. Show that $\mathcal{S}_2$ can be partitioned into four triangular regions, on each of which f is differentiable. In other words, find the four formulas for $f(2, y, z)$, and describe the domain of each.

403. (Continuation) Show that $f(E) = \frac{1}{4}\sqrt{130}$, for a unique point $E = (2, y, z)$ on $\mathcal{S}_2$. This is the maximal value of f.

404. (Continuation) Describe the appearance of the level curves of f restricted to $\mathcal{S}_2$.

405. Consider the rectangular box $\mathcal{B}$ in $\mathbf{R}^3$, defined by the three simultaneous inequalities $0 \le x \le 5/3$, $0 \le y \le 1$, and $0 \le z \le 1$, and let $\mathcal{S} = \partial\mathcal{B}$ be its boundary surface. For any point P on $\mathcal{S}$, let $f(P)$ be the distance (measured along $\mathcal{S}$) from $O = (0,0,0)$ to P. Verify that $f(C_3) = \frac{1}{3}\sqrt{61}$, where $C_3 = (5/3, 1, 1)$.

406. (Continuation) Let $\mathcal{S}_2$ be the square face of $\partial\mathcal{B}$ on which $x = 5/3$. Show that $\mathcal{S}_2$ can be partitioned into four triangular regions, on each of which f is differentiable. In other words, find the four formulas for $f(5/3, y, z)$, and describe the domain of each.

407. (Continuation) Calculate $f(E)$, where $E = \left(\dfrac{5}{3}, \dfrac{4}{5}, \dfrac{4}{5}\right)$.

408. (Continuation) Calculate $f(S)$, where $S = \left(\dfrac{5}{3}, \dfrac{5}{6}, \dfrac{5}{6}\right)$.

409. (Continuation) Show that $f(C_3)$ and $f(E)$ are locally maximal.

410. (Continuation) Show that f is not differentiable at any point on the diagonal $y = z$ of square face $\mathcal{S}_2$.

411. (Continuation) In spite of this result, make a case for S being a saddle point for f.

412. (Continuation) Describe the appearance of the level curves of f restricted to $\mathcal{S}_2$.

Inverse Square Law Orbits

413. One of the following equations can be solved for $\mathbf{u}$. Find the equation, then find at least two different solutions for it:
(a) $[3, 1, -2] \times \mathbf{u} = [2, -1, 2]$ **(b)** $[3, 1, -2] \times \mathbf{u} = [1, 1, 2]$ **(c)** $[3, 1, -2] \times \mathbf{u} = [3, 1, -2]$

414. The vectors

$$\begin{aligned}\mathbf{u} &= [\cos\theta, \sin\theta]\\ \mathbf{u}_\theta &= [-\sin\theta, \cos\theta]\\ \frac{d\mathbf{u}_\theta}{d\theta} &= [-\cos\theta, -\sin\theta]\end{aligned}$$

are useful when using polar coordinates to analyze plane curves. In the context of vector dynamics (for example, Kepler's Laws), the notation $\mathbf{u}_\theta{}'$ means the derivative, with respect to t, of the unit vector $\mathbf{u}_\theta$. It could be written less ambiguously as $\dfrac{d\mathbf{u}_\theta}{dt}$. Express this vector in terms of $\mathbf{u}$.

415. Given a path $\mathbf{r} = r\mathbf{u}$, where $\mathbf{r}$ is the radial vector from the origin in the xy-plane, you have calculated

$$\mathbf{r}'' = \left(\frac{d^2r}{dt^2} - r\frac{d\theta}{dt}\frac{d\theta}{dt}\right)\mathbf{u} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{u}_\theta.$$

by using the Product Rule and the Chain Rule. In order for $\mathbf{r}$ to be governed by a central force, it is therefore necessary that the coefficient of $\mathbf{u}_\theta$ be zero:

$$0 = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}$$

If $\mathbf{r}''$ is produced by a central force, this should be guaranteed by the area-sweeping fact $r^2\theta' = h$ derived earlier. Verify that it is.

416. (Continuation) Suppose that the inverse-square law $\mathbf{r}'' = -\dfrac{g}{r^2}\,\mathbf{u}$ applies. Derive the equation $r'' = \dfrac{h^2 - gr}{r^3}$ by comparing coefficients of $\mathbf{u}$.

417. (Continuation) This *second-order* differential equation describes r implicitly as a function of t. This is a difficult equation to solve. First, a couple of easy steps:
(a) There is one solution $r(t) = r_0$, where r_0 is constant. Find the constant, and then find a corresponding angular function $\theta(t)$.
(b) Because r'' is expressed in terms of r, it is reasonable to expect that r' should also be expressible in terms of r. Show that any r that satisfies $(r')^2 = \dfrac{2g}{r} + C - \dfrac{h^2}{r^2}$, where C is a constant, will also satisfy the equation for r''.

418. Under the influence of an inverse-square law $\mathbf{r}'' = -\dfrac{g}{r^2}\,\mathbf{u}$, the position of an object fits an equation $(r')^2 + \dfrac{h^2}{r^2} = \dfrac{2g}{r} + C$, where C is a constant. (If you know physics, you should verify that C is proportional to the total energy of the object.) If the constant C happens to be *negative*, then the values of r are bounded — that is, they cannot become arbitrarily large. Explain why. In fact, show that r must always be smaller than $-2g/C$. This estimate will soon be improved.

419. The differential equation can be written as $r^2(r')^2 = 2gr + Cr^2 - h^2$. If the constant C is negative, the resulting elliptical orbit has a minimal value and a maximal value for r, which occur at the vertices of the ellipse. At either point, r' is zero. Why? Use this information to show that the minimal value of r (at *perihelion*) is

$$r_0 = \frac{-g + \sqrt{g^2 + Ch^2}}{C}$$

and the maximal value of r (at *aphelion*) is

$$r_1 = \frac{-g - \sqrt{g^2 + Ch^2}}{C}$$

This formula for r_1 improves an earlier estimate. How? (Remember that C is negative.)

420. It should be clear that $2gr + Cr^2 - h^2 = 0$ when $r = r_0$ and when $r = r_1$. What are the values of $\dfrac{g + Cr}{\sqrt{g^2 + Ch^2}}$ when $r = r_0$ and when $r = r_1$? They will be useful below.

421. Temporarily write the negative value of C as $-k^2$, where k is positive. Verify that the differential equation can be rewritten equivalently as $r^2(r')^2 = 2gr - k^2r^2 - h^2$. Now complete the square to obtain the version

$$\frac{r^2(r')^2}{\dfrac{g^2}{k^2} - h^2 - \left(kr - \dfrac{g}{k}\right)^2} = 1.$$

422. Assume now that r' is nonnegative (the planet is receding from the Sun). Show how the differential equation can be rewritten as

$$\frac{\left(kr - \dfrac{g}{k}\right) \cdot kr' + gr'}{\sqrt{\dfrac{g^2}{k^2} - h^2 - \left(kr - \dfrac{g}{k}\right)^2}} = k^2.$$

423. How would the preceding equation have looked if we had assumed that $r' \le 0$?

424. Assuming that $0 \le r'$, antidifferentiate both sides of the differential equation to obtain

$$-\sqrt{\frac{g^2}{k^2} - h^2 - \left(kr - \frac{g}{k}\right)^2} + \frac{g}{k}\,\sin^{-1}\left(\frac{kr - \dfrac{g}{k}}{\sqrt{\dfrac{g^2}{k^2} - h^2}}\right) = k^2(t - t_0),$$

in which t_0 serves as the antidifferentiation constant (its value will be chosen soon). Now restore C to obtain the equation

$$-\sqrt{2gr + Cr^2 - h^2} - \frac{g}{\sqrt{-C}}\,\sin^{-1}\left(\frac{g + Cr}{\sqrt{g^2 + Ch^2}}\right) = -C(t - t_0),$$

which implicitly defines r as a function of t. It is clear that there is no chance of solving *explicitly* for r in terms of t. It is also clear that this equation does not completely describe the periodic nature of the orbit, because the equation is valid only for a restricted time interval. The rearrangement

$$\sin\left(\frac{\sqrt{-C}}{g}\,\sqrt{2gr + Cr^2 - h^2} - \frac{C\sqrt{-C}}{g}\,(t - t_0)\right) + \frac{g + Cr}{\sqrt{g^2 + Ch^2}} = 0\,.$$

is better, but it does not correctly describe the half of the orbit where $r' \le 0$.

425. It is convenient to adopt ω as an abbreviation for $\dfrac{-C\sqrt{-C}}{g}$. Notice that inserting $r = r_0$ into the preceding equation reduces the equation to $\sin(\omega(t - t_0)) = -1$. Conclude that the planet passes perihelion at $t = 0$ and aphelion at $t = \dfrac{\pi}{\omega}$, if t_0 is chosen to be $\dfrac{\pi}{2\omega}$.

426. Using this choice for t_0, show that

$$\cos\left(\omega t - \frac{\omega}{C}\,\sqrt{2gr + Cr^2 - h^2}\right) = \frac{g + Cr}{\sqrt{g^2 + Ch^2}}$$

describes the planet's orbit for $0 \le t \le \dfrac{\pi}{\omega}$.

427. By modifying the preceding in the case $r' \le 0$, show that

$$\cos\left(\omega t + \frac{\omega}{C}\,\sqrt{2gr + Cr^2 - h^2}\right) = \frac{g + Cr}{\sqrt{g^2 + Ch^2}}$$

describes the planet's orbit for $\dfrac{\pi}{\omega} \le t \le \dfrac{2\pi}{\omega}$.

428. By combining the two preceding cases, you can show that the equation

$$\cos\left(\left|2\sin^{-1}\left(\sin\left(\frac{1}{2}\,\omega t\right)\right)\right| - \frac{\omega}{C}\,\sqrt{2gr + Cr^2 - h^2}\right) = \frac{g + Cr}{\sqrt{g^2 + Ch^2}}$$

describes the planet's orbit for *all* values of t.

429. Once r is known as a function of t, the ellipse equation provides a quick way to obtain θ as a function of t. Because r is known only implicitly, however, finding an explicit formula for θ seems to be out of the question as well. Show that θ is defined implicitly by the equation $\cos\theta = \dfrac{h^2 - gr}{r\sqrt{g^2 + Ch^2}}$, assuming that the planet passes perihelion at $t = 0$.

430. Because C is negative, you might be wondering whether the constant $\sqrt{g^2 + Ch^2}$ is well-defined. Show that it is equal to $r_0v_0^2 - g$, where v_0 is the speed at perihelion.

431. You are perhaps wondering whether $\sqrt{2gr + Cr^2 - h^2}$ is well-defined, given that the positive term $2gr$ is diminished by two negative terms. First show that

$$\sqrt{2gr + Cr^2 - h^2} = \sqrt{2gr\left(1 - \frac{r}{r_0}\right) + (r^2 - r_0^2)v_0^2}\,.$$

Because $r_0 \le r$, you are dealing with a sum of two terms, one of which is never positive, the other of which is never negative. Because $r \le r_1$, the sum is nonnegative. Prove these remarks.

432. It is intuitively clear that an orbit is completely determined by the two numbers r_0 (minimial radius) and $v_0 = r_0\theta_0'$ (maximal speed) at perihelion. Except for g (the only true constant), the values of other “constants” are determined by r_0 and v_0. It should therefore be possible to express each of them in terms of r_0 and v_0. Confirm each of the following formulas:

(h) The “sweeping” constant h equals r_0v_0.

(c) The “energy” constant C equals $v_0^2 - \dfrac{2g}{r_0}$.

(m) The eccentricity m equals $\dfrac{r_0v_0^2}{g} - 1$.

(d) The mean distance $\mathcal{D}$ to the Sun equals $\dfrac{r_0 g}{2g - r_0v_0^2}$.

(t) The period $\mathcal{T}$ equals $2\pi g\left(\dfrac{r_0}{2g - r_0v_0^2}\right)^{3/2}$.

(r) The maximal radius r_1 equals $\dfrac{r_0^2v_0^2}{2g - r_0v_0^2}$.

433. The type of orbit depends on the numerical data r_0 and v_0 at perihelion. Use the formula for eccentricity to explain why

(a) the orbit is a circle if $g = r_0v_0^2$;

(b) the orbit is an ellipse if $g < r_0v_0^2 < 2g$;

(c) the orbit is a parabola if $2g = r_0v_0^2$;

(d) the orbit is a hyperbola if $2g < r_0v_0^2$;

(e) the case $0 < r_0v_0^2 < g$ must be excluded from this summary.

434. It is intuitively clear that an orbit is determined by the two numbers C and h. You have already shown that r_0 and r_1 — the extreme values of r for an elliptical orbit — can be expressed in terms of C and h (and g). Show also that

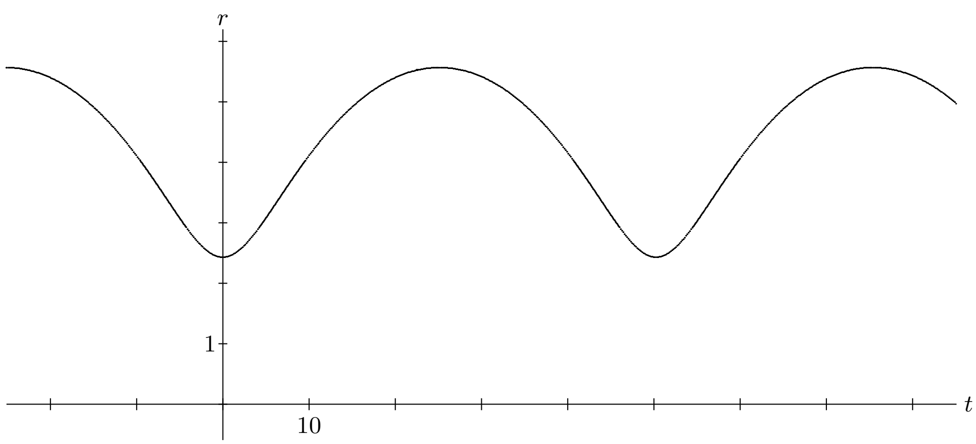
(v) the maximal speed v_0 equals $\dfrac{g + \sqrt{g^2 + Ch^2}}{h}$;

(d) the mean distance $\mathcal{D}$ equals $-\dfrac{g}{C}$;

(t) the period $\mathcal{T}$ equals $2\pi g(-C)^{-3/2}$;

(m) the eccentricity m equals $\dfrac{\sqrt{g^2 + Ch^2}}{g}$.

435. The diagram shows the graph of $r(t)$ that corresponds to the data $g = 1$, $C = -0.25$, and $h = 1.84$, with perihelion occurring at $t = 0$. Use these numbers to calculate the mean distance $\mathcal{D}$, the period $\mathcal{T}$, and the extreme radial values r_0 and r_1. Use the graph to check your answers. Also calculate the eccentricity of the elliptical orbit.



436. Use the data for the graph above to calculate the extreme values v_0 and v_1 of the orbital speed. Explain how the graph above reinforces the result that v_1 is significantly smaller than v_0.

437. The graph also should make it clear that $\mathcal{D}$ — the so-called "mean distance" — is not the average value of r with respect to *time*. Use the graph to estimate the value of this average. You will soon be able to calculate it exactly.

438. The above formula expresses r as a function of the polar angle θ. According to this description, the average value of the focal radius r is

$$\overline{r} = \frac{1}{2\pi}\int_0^{2\pi} r\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{b^2}{a + c\cos\theta}\,d\theta.$$

It is not a trivial exercise to evaluate this integral by finding an antiderivative, but it can be done. It is also interesting to see what your calculator can come up with. Whatever your method, find evidence that $\overline{r} = b$. Remember that $a^2 = b^2 + c^2$.

439. It is known that planets move around their elliptical orbits (which have the sun at one focus) with nonconstant speed, going more slowly when they are farthest from the sun. In fact, Kepler's Third Law says that

$$\frac{d\theta}{dt} = \frac{h}{r^2},$$

where h is a constant and t is time. This enables us to express the length of the planetary year as

$$T = \int_0^{2\pi} \frac{r^2}{h}\,d\theta.$$

According to this point of view, the average value of r is

$$\overline{r} = \frac{1}{T}\int_0^{T} r\,dt = \frac{1}{T}\int_0^{2\pi} \frac{r^3}{h}\,d\theta.$$

In other words, the average value of r is $\displaystyle\int_0^{2\pi} r^3\,d\theta$ divided by $\displaystyle\int_0^{2\pi} r^2\,d\theta$, because h is constant. As above, these are challenging integrals to do by means of antiderivatives. Whatever your method, find evidence that the values of the integrals are $2\pi(a^2b + \frac{1}{2}c^2b)$ and $2\pi ab$, respectively. Does the value for $\overline{r}$ agree with the value found in item 2?

440. By viewing r as a function of x for $-a - c \le x \le a - c$, obtain a as a plausible value for $\overline{r}$. (*Hint*: r is in fact a linear function of x.)

441. In the MKS system, the inverse-square constant for acceleration vectors directed toward the Sun is $g = 1.328\times10^{20}$. (The units are meters^3/seconds^2.) The mean distance $\mathcal{D}$ between the Earth and the Sun is 1.496×10^{11} meters, and the eccentricity of the Earth's orbit is 0.0167. Use these values to calculate **(a)** the smallest distance r_0 between the Earth and the Sun; **(b)** how much time passes between one perihelion and the next; **(c)** the Earth's largest orbital speed.

Miscellaneous Problems

442. Let $p(x)$ be a polynomial function whose values have a *lower bound*. This means that there is a number m with the property $m \le p(x)$ for all real numbers x. Show that p attains its global minimum, by showing that one of its values is a lower bound

443. (Continuation) State and prove a version for *upper bounds*.

444. Show that the preceding statements are false for non-polynomial functions.

445. Let $p(x, y) = x^2 - 2x + y^2 + 4y + 5$. Show that this polynomial has lower bounds. Does p attain its global minimum? Explain.

446. Let $p(x, y)$ be a polynomial function whose values have a lower bound. Is it necessarily true that p attains its global minimum? If so, prove it. If not, invent a counterexample.

Mathematics 6 Reference

absolutely convergent: A series Σa_n for which $\Sigma|a_n|$ converges. In other words, Σa_n converges, regardless of the pattern of its signs.

acceleration: The derivative of velocity with respect to time.

alternating series: A *series* of real numbers in which every other term is positive.

AM-GM inequality: The *geometric mean* never exceeds the *arithmetic mean*, and the two agree only when all the numbers are equal.

angle-addition identities: For any angles α and β, $\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

angle between vectors: When two vectors **u** and **v** are placed tail-to-tail, the angle θ they form can be calculated by the dot-product formula $\cos\theta = \dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$. If $\mathbf{u}\bullet\mathbf{v} = 0$ then **u** is perpendicular to **v**. If $\mathbf{u}\bullet\mathbf{v} < 0$ then **u** and **v** form an obtuse angle.

antiderivative: If f is the derivative of g, then g is called *an antiderivative of f*. For example, $g(x) = 2x\sqrt{x} + 5$ is an antiderivative of $f(x) = 3\sqrt{x}$, because $g' = f$.

aphelion: The point on an orbit that is farthest from the attracting focus.

arc length: A common application of integration.

arccos: This is another name for the inverse cosine function, commonly denoted $\cos^{-1}$.

arcsin: This is another name for the inverse sine function, commonly denoted $\sin^{-1}$.

arctan: This is another name for the inverse tangent function, commonly denoted $\tan^{-1}$.

arithmetic mean: If n numbers are given, their arithmetic mean is the sum of the numbers, divided by n.

asteroid: Do not confuse a small, planet-like member of our solar system with an *astroid*.

astroid: A type of cycloid, this curve is traced by a point on a wheel that rolls without slipping around the inside of a circle whose radius is four times the radius of the wheel. First mentioned by Leibniz, in 1715.

average value: If $f(x)$ is defined on an interval $a \le x \le b$, the average of the values of f on this interval is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. If $f(x,y)$ is defined on a region $\mathcal{R}$, the average of the values of f on this region is $\dfrac{1}{\text{area}(\mathcal{R})}\displaystyle\int_{\mathcal{R}} f(x,y)\,dx\,dy$. If $f(x,y,z)$ is defined on a region $\mathcal{V}$, the average of the values of f on this region is $\dfrac{1}{\text{volume}(\mathcal{V})}\displaystyle\int_{\mathcal{V}} f(x,y,z)\,dx\,dy\,dz$.

average velocity is displacement divided by elapsed time.

Mathematics 6 Reference

bounded: Any subset of $\mathbf{R}^n$ that is contained in a suitably large *disk*.

cardioid: A *cycloid*, traced by a point on a circular wheel that rolls without slipping around another circular wheel of the same size.
catenary: Modeled by the graph of the *hyperbolic function* cosh, this is the shape assumed by a hanging chain.

center of curvature: Given points $\mathbf{p}$ and $\mathbf{q}$ on a differentiable plane curve, let $\mathbf{c}$ be the intersection of the lines *normal* to the curve at $\mathbf{p}$ and $\mathbf{q}$. The limiting position of $\mathbf{c}$ as $\mathbf{q}$ approaches $\mathbf{p}$ is the center of curvature of the curve at $\mathbf{p}$. For non-planar curves, there are many normal lines from which to choose, so an "instantaneous" plane must be specified. One way to select the principal normal direction is to define it as the derivative of the unit tangent vector.

central force: A spherically symmetric vector field.

centroid of a region: Of all the points in the region, this is their average.

centroid of an arc: Of all the points on the arc, this is their average.

Chain Rule: The derivative of a composite function $C(x) = f(g(x))$ is a product of derivatives, namely $C'(x) = f'(g(x))g'(x)$. The actual appearance of this rule changes from one example to another, because of the variety of function types that can be composed. For example, a curve can be traced in $\mathbf{R}^3$, on which a real-valued temperature distribution is given; the composite $\mathbf{R}^1 \longrightarrow \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ simply expresses temperature as a function of time, and the derivative of this function is the dot product of two vectors.

chord: A segment that joins two points on a curve.

cis θ: Stands for the unit complex number $\cos\theta + i\sin\theta$. Also known as $e^{i\theta}$.

closed: Suppose that $\mathcal{D}$ is a set of points in $\mathbf{R}^n$, and that every convergent sequence of points in $\mathcal{D}$ actually converges to a point in $\mathcal{D}$. Then $\mathcal{D}$ is called "closed."

comparison of series: Given two infinite series Σa_n and Σb_n, about which $0 < a_n \le b_n$ is known to be true for all n, the convergence of Σb_n implies the convergence of Σa_n, and the divergence of Σa_n implies the divergence of Σb_n.

concavity: A graph $y = f(x)$ is *concave up* on an interval if f'' is positive throughout the interval. The graph is *concave down* on an interval if f'' is negative throughout the interval.

conditionally convergent: A convergent series Σa_n for which $\Sigma|a_n|$ diverges.

conic section: Any graph obtainable by slicing a cone with a cutting plane. This might be an ellipse, a parabola, a hyperbola, or some other special case.

content: A technical term that is intended to generalize the special cases length, area, and volume, so that the word can be applied in any dimension.

continuity: A function f is *continuous at a* if $f(a) = \lim_{p \to a} f(p)$. A *continuous function* is continuous at all the points in its domain.

converge (sequence): If the terms of a *sequence* come arbitrarily close to a fixed value, the sequence is said to *converge* to that value.

converge (series): If the *partial sums* of an infinite *series* come arbitrarily close to a fixed value, the series is said to *converge* to that value.

converge (integral): An *improper integral* that has a finite value is said to *converge* to that value, which is defined using a limit of proper integrals.

cosh: See *hyperbolic functions*.

critical point: A point in the domain of a function f at which f' is either zero or undefined.

cross product: Given $\mathbf{u} = [p, q, r]$ and $\mathbf{v} = [d, e, f]$, a vector that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is $[\,qf - re, rd - pf, pe - qd\,] = \mathbf{u} \times \mathbf{v}$.

curl: A three-dimensional vector field that describes the rotational tendencies of the three-dimensional field from which it is derived.

curvature: This positive quantity is the rate at which the direction of a curve is changing, with respect to the distance traveled along it. For a circle, this is just the reciprocal of the radius. The principal *normal vector* points towards the center of curvature.

cycloid: A curve traced by a point on a wheel that rolls without slipping. Galileo named the curve, and Torricelli was the first to find its area.

cylindrical coordinates: A three-dimensional system of coordinates obtained by appending z to the usual polar-coordinate pair (r, θ).

decreasing: A function f is *decreasing* on an interval $a \le x \le b$ if $f(v) < f(u)$ holds whenever $a \le u < v \le b$ does.

derivative: Let f be a function that is defined for points $\mathbf{p}$ in $\mathbf{R}^n$, and whose values $f(\mathbf{p})$ are in $\mathbf{R}^m$. If it exists, the derivative $f'(\mathbf{a})$ is the $m \times n$ matrix that represents the best possible linear approximation to f at $\mathbf{a}$. In the case $n = 1$ (a parametrized curve in $\mathbf{R}^m$), $f'(a)$ is the $m \times 1$ matrix that is visualized as the tangent vector at $f(a)$. In the case $m = 1$, the $1 \times n$ matrix $f'(\mathbf{a})$ is visualized as the gradient vector at $\mathbf{a}$.

Mathematics 6 Reference

derivative at a point: Let f be a real-valued function that is defined for points in $\mathbf{R}^n$. Differentiability at a point $\mathbf{a}$ in the domain of f means that there is a linear function L with the property that the difference between $L(\mathbf{p})$ and $f(\mathbf{p})$ approaches 0 faster than $\mathbf{p}$ approaches $\mathbf{a}$, meaning that $0 = \lim_{\mathbf{p}\to\mathbf{a}} \dfrac{f(\mathbf{p}) - L(\mathbf{p})}{|\mathbf{p}-\mathbf{a}|}$. If such an L exists, then $f'(\mathbf{a})$ is the matrix that defines $L(\mathbf{p}-\mathbf{a})$.

determinant: A ratio that is associated with any square matrix, as follows: Except for a possible sign, the determinant of a 2×2 matrix $\mathbf{M}$ is the area of any region $\mathcal{R}$ in 2-dimensional space, divided into the area of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$. Except for a possible sign, the determinant of a 3×3 matrix $\mathbf{M}$ is the volume of any region $\mathcal{R}$ in 3-dimensional space, divided into the volume of the region that results when $\mathbf{M}$ is applied to $\mathcal{R}$.

differentiable: A function that has derivatives at all the points in its domain.

differential equation: An equation that is expressed in terms of an unknown function and its derivative. A solution to a differential equation is a function.

differentials: Things like dx, dt, and dy. Called "ghosts of departed quantities" by George Berkeley (1685-1753), who was skeptical of Newton's approach to mathematics.

directional derivative: Given a function f defined at a point $\mathbf{p}$ in $\mathbf{R}^n$, and given a direction $\mathbf{u}$ (a unit vector) in $\mathbf{R}^n$, the derivative $D_{\mathbf{u}}f(\mathbf{p})$ is the instantaneous rate at which the values of f change when the input varies only in the direction specified by $\mathbf{u}$.

discontinuous: A function f has a *discontinuity at a* if $f(a)$ is defined but does not equal $\lim_{p\to a} f(p)$; a function is *discontinuous* if it has one or more discontinuities.

disk: Given a point $\mathbf{c}$ in $\mathbf{R}^n$, the set of all points $\mathbf{p}$ for which the distance $|\mathbf{p}-\mathbf{c}|$ is at most r is called the disk (or "ball") of radius r, centered at $\mathbf{c}$.

diverge means *does not converge*.

divergence: If $\mathbf{v}$ is a vector field, its divergence is the scalar function $\nabla \bullet \mathbf{v}$.

domain: The domain of a function consists of all the numbers for which the function returns a value. For example, the domain of a logarithm function consists of positive numbers only.

double-angle identities: Best-known are $\sin 2\theta \equiv 2\sin\theta\cos\theta$, $\cos 2\theta \equiv 2\cos^2\theta - 1$, and $\cos 2\theta \equiv 1 - 2\sin^2\theta$; special cases of the *angle-addition identities*.

Mathematics 6 Reference

double integral: A descriptive name for an integral whose domain of integration is two-dimensional. When possible, evaluation is an iterative process, whereby two single-variable integrals are evaluated instead.

e is approximately 2.71828182845904523536. This irrational number frequently appears in scientific investigations. One of the many ways of defining it is $e = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

eccentricity: For curves defined by a focus and a directrix, this number determines the shape of the curve. It is the distance to the focus divided by the distance to the directrix, measured from any point on the curve. The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle (a special ellipse) is 0. The word is pronounced "eck-sen-trissity".

ellipse I: An ellipse is determined by a focal point, a directing line, and an eccentricity between 0 and 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity.

ellipse II: An ellipse has two focal points. The sum of the *focal radii* to any point on the ellipse is constant.

ellipsoid: A *quadric surface*, all of whose planar sections are ellipses.

equiangular spiral: The angle formed by the radius vector and the tangent vector is the same at every point on the spiral. Any polar curve of the form $r = b^\theta$ has this propery.

Euler's Method: Given a differential equation, a starting point, and a step size, this method provides an approximate numerical solution to the equation. Leonhard Euler (1707-1783) was a prolific Swiss mathematician who did much of his work while blind. He was the first to find the exact value of the convergent series $\sum_{n=1}^{\infty} n^{-2}$. He had 13 children.

evolute: Given a differentiable curve $\mathcal{C}$, this is the configuration of centers of curvature for $\mathcal{C}$.

Extended Mean-Value Theorem: If f is a function that is $n+1$ times differentiable for $0 \le x \le b$, then

$$f(b) = f(0) + f'(0)\,b + \frac{1}{2}f''(0)\,b^2 + \ \cdots\ + \frac{1}{n!}f^{(n)}(0)\,b^n + \frac{1}{(n+1)!}f^{(n+1)}(c)\,b^{n+1}$$

for some c between 0 and b. This version of the theorem is due to *Lagrange*.

extreme point: either a *local minimum* or a *local maximum*. Also called an *extremum*.

Mathematics 6 Reference

Extreme-value Theorem: Suppose that f is a continuous real-valued function that is defined throughout a *closed* and *bounded* set $\mathcal{D}$ of points. Then f attains a maximal value and a minimal value on $\mathcal{D}$. This means that there are points $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{D}$, such that $f(\mathbf{a}) \le f(\mathbf{p}) \le f(\mathbf{b})$ holds for all $\mathbf{p}$ in $\mathcal{D}$. If f is also differentiable, then $\mathbf{a}$ is either a critical point for f, or it belongs to the boundary of $\mathcal{D}$; the same is true of $\mathbf{b}$.

focal radius: A segment that joins a point on a conic section to one of the focal points; also used to indicate the length of such a segment.

Fubini's Theorem: Provides conditions under which the value of an integral is independent of the iterative approach applied to it.

Fundamental Theorem of Algebra: Every complex polynomial of degree n can be factored (in essentially only one way) into n linear factors.

Fundamental Theorem of Calculus: In its narrowest sense, differentiation and integration are inverse procedures — integrating a derivative $f'(x)$ along an interval $a \le x \le b$ leads to the same value as forming the difference $f(b) - f(a)$. In multivariable calculus, this concept evolves.

geometric mean: If n positive numbers are given, their geometric mean is the n^{th} root of their product.

geometric sequence: A list in which each term is obtained by applying a constant multiplier to the preceding term.

geometric series: An infinite example takes the form $a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$. Such a series converges if, and only if, $|r| < 1$, in which case its sum is $\dfrac{a}{1-r}$.

global maximum: Given a function f, this may or may not exist. It is the value $f(c)$ that satisfies $f(x) \le f(c)$ for all x in the domain of f.

global minimum: Given a function f, this may or may not exist. It is the value $f(c)$ that satisfies $f(c) \le f(x)$ for all x in the domain of f.

gradient: This is the customary name for the *derivative* of a real-valued function, especially when the domain is multidimensional.

Greek letters: Apparently essential for doing serious math! There are 24 letters. The upper-case characters are

$$\mathrm{A\ B}\ \Gamma\ \Delta\ \mathrm{E\ Z\ H}\ \Theta\ \mathrm{I\ K}\ \Lambda\ \mathrm{M\ N}\ \Xi\ \mathrm{O}\ \Pi\ \mathrm{P}\ \Sigma\ \mathrm{T}\ \Upsilon\ \Phi\ \mathrm{X}\ \Psi\ \Omega$$

and the corresponding lower-case characters are

$$\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \zeta\ \eta\ \theta\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \xi\ o\ \pi\ \rho\ \sigma\ \tau\ \upsilon\ \phi\ \chi\ \psi\ \omega$$

Green's Theorem: Equates a given *line integral* to a special double integral over the region enclosed by the given contour. The self-taught George Green (1793-1841) developed a mathematical theory of electricity and magnetism.

Gregory's Series: The alternating sum of the reciprocals of odd integers is a convergent infinite series. Its sum is $\frac{1}{4}\pi$.

Hadamard: See *Prime Number Theorem.*

harmonic series: The sum of the reciprocals of the positive integers.

heat equation: A partial differential equation that describes the conduction of heat.

Heaviside operator: The use of a symbol, such as D or D_x, to indicate the differentiation process. The scientist Oliver Heaviside (1850-1925) advocated the use of vector methods, clarified Maxwell's equations, and introduced operator notation so that solving differential equations would become a workout in algebra.

Hessian: See *second derivative.*

hyperbola I: A hyperbola has two focal points, and the difference between the *focal radii* drawn to any point on the hyperbola is constant.

hyperbola II: A hyperbola is determined by a focal point, a directing line, and an eccentricity greater than 1. Measured from any point on the curve, the distance to the focus divided by the distance to the directrix is always equal to the eccentricity.

hyperbolic functions: Just as the properties of the circular functions sin, cos, and tan are consequences of their definition using the unit circle $x^2 + y^2 = 1$, the analogous properties of sinh, cosh, and tanh follow from their definition using the unit hyperbola $x^2 - y^2 = 1$.

hyperboloid: One of the *quadric surfaces.* Its principal plane of reflective symmetry has a special property — every section obtained by slicing the surface perpendicular to this plane is a hyperbola.

implicit differentiation: Applying a differentiation operator to an identity that has not yet been solved for a dependent variable in terms of its independent variable.

improper integral: This is an integral $\int_{\mathcal{D}} f$ for which the domain $\mathcal{D}$ of integration is unbounded, or for which the values of the integrand f are undefined or unbounded.

increasing: A function f is *increasing* on an interval $a \le x \le b$ if $f(u) < f(v)$ holds whenever $a \le u < v \le b$ does.

indeterminate form: This is an ambiguous limit expression, whose actual value can only be deduced by looking at the given example. The five most common types are:

$\dfrac{0}{0}$, examples of which are $\displaystyle\lim_{t\to 0}\frac{\sin t}{t}$ and $\displaystyle\lim_{h\to 0}\frac{2^h-1}{h}$

1^∞, examples of which are $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

$\dfrac{\infty}{\infty}$, examples of which are $\displaystyle\lim_{x\to\infty}\frac{2x+1}{3x+5}$ and $\displaystyle\lim_{x\to 0}\frac{\log_2 x}{\log_3 x}$

$0\cdot\infty$, examples of which are $\displaystyle\lim_{x\to 0} x\ln x$ and $\displaystyle\lim_{x\to\pi/2}\left(x-\frac{1}{2}\pi\right)\tan x$

$\infty-\infty$, examples being $\displaystyle\lim_{x\to\infty}\sqrt{x^2+4x}-x$ and $\displaystyle\lim_{x\to\pi/2}\sec x\tan x-\sec^2 x$

The preceding limit examples all have different values.

infinite series: To find the sum of one of these, you must look at the limit of its partial sums. If the limit exists, the series *converges*; otherwise, it *diverges*.

inflection point: A point on a graph $y=f(x)$ where f'' changes sign.

integral test: A method of establishing convergence for positive, decreasing series of terms, by comparing them with improper integrals.

integrand: A function whose integral is requested.

interval of convergence: Given a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, the x-values for which the series (absolutely) converges form an interval, by the *Ratio Test*. For example, the *geometric series* $\displaystyle\sum_{n=0}^{\infty} x^n$ converges for $-1<x<1$. Also see *radius of convergence*.

inverse function: Any function f processes input values to obtain output values. A function that undoes what f does is said to be *inverse* to f, and often denoted f^{-1}. In other words, $f^{-1}(b)=a$ must hold whenever $f(a)=b$ does. For some functions ($f(x)=x^2$, for example), it is necessary to restrict the domain in order to define an inverse.

involute: Given a point on a differentiable curve (whose curvature is of constant sign), an involute is defined by imagining the point to be the end of a thread that initially coincides with the curve, and which is then "unwound" from the curve while keeping the thread taut; the involute is the trace of the free end.

isocline: A curve, all of whose points are assigned the same slope by a differential equation.

isotherm: A special case of the level-curve or level-surface concept.

Jacobian: A traditional name for the derivative of a function f from $\mathbf{R}^n$ to $\mathbf{R}^m$. For each point $\mathbf{p}$ in the domain space, $f'(\mathbf{p})$ is an $m \times n$ matrix. When $m = n$, the matrix is square, and its determinant is also called "the Jacobian" of f. Carl Gustav Jacobi (1804-1851) was a prolific mathematician; one of his lesser accomplishments was to establish the symbol ∂ for partial differentiation.

Kepler's First Law: Planets travel in elliptical orbits, with the Sun at one focus.

Kepler's Second Law: The vector that points from the Sun to a planet sweeps out area at a constant rate.

Kepler's Third Law: Divide the cube of the mean distance from a planet to the Sun by the square of the time it takes for the planet to complete its orbit around the Sun — the result is the same number k for every planet. The ratio depends only on the units used in the calculation. In other words, $d^3 = kt^2$. If distances are expressed in astronomical units, k equals 1. The theory applies equally well to the satellites of a planet. Johannes Kepler (1571-1630) supported his astronomical publications by selling astrological calendars.

l'Hôpital's Rule: A method for dealing with indeterminate forms: If f and g are differentiable, and $f(a) = 0 = g(a)$, then $\lim\limits_{t \to a} \dfrac{f(t)}{g(t)}$ equals $\lim\limits_{t \to a} \dfrac{f'(t)}{g'(t)}$, provided that the latter limit exists. The Marquis de l'Hôpital (1661-1704) wrote the first textbook on calculus.

Lagrange multipliers: A method for solving constrained extreme-value problems.

Lagrange's error formula: Given a function f and one of its Taylor polynomials p_n based at $x = a$, the difference between $f(x)$ and $p_n(x)$ is $\dfrac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$, for some c that is between a and x. Joseph Lagrange (1736-1813) made many contributions to calculus and analytic geometry, including a simple notation for derivatives.

Lagrange notation: The use of primes to indicate derivatives.

lemniscate: Given two focal points that are separated by a distance $2c$, the lemniscate consists of points for which the product of the focal radii is c^2.

level curve: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^2$ and k is a constant.

level surface: The configuration of points $\mathbf{p}$ that satisfy an equation $f(\mathbf{p}) = k$, where f is a real-valued function defined for points in $\mathbf{R}^3$ and k is a constant.

limaçon: This cycloidal curve is traced by an arm of length $2r$ attached to a wheel of radius r that is rolling around a circle of the same size.

line integral: Given a *vector field* F and a *path* $\mathcal{C}$ (which does not have to be linear) in the domain space, a real number results from "integrating F along $\mathcal{C}$".

local maximum: Given a function f, this is a value $f(c)$ that satisfies $f(x) \le f(c)$ for all x in some suitably small interval containing c.

local minimum: Given a function f, this is a value $f(c)$ that satisfies $f(c) \le f(x)$ for all x in some suitably small interval containing c.

logarithmic integral: See *Hadamard*.

logarithmic spiral: A curve described in polar coordinates by an equation $r = a \cdot b^{\theta}$.

Maclaurin polynomials: Given a highly differentiable function, the values of its derivatives at $x = 0$ are used to create these ideal approximating polynomials. They can be viewed as the partial sums of the Maclaurin *series* for the given function. Colin Maclaurin (1698-1746) wrote papers about calculus and analytic geometry. He learned about Maclaurin series from the writings of Taylor and Stirling.

Mean-Value Theorem: If the curve $y = f(x)$ is continuous for $a \le x \le b$, and differentiable for $a < x < b$, then the slope of the line through $(a, f(a))$ and $(b, f(b))$ equals $f'(c)$, where c is strictly between a and b. There is also a version of this statement that applies to integrals.

moment: Quantifies the effect of a force that is magnified by applying it to a lever. Multiply the length of the lever by the magnitude of the force.

normal vector: In general, this is a vector that is perpendicular to something (a line or a plane). In the analysis of parametrically defined curves, the principal normal vector (which points in the direction of the center of curvature) is the derivative of the unit tangent vector.

odd function: A function whose graph has half-turn symmetry at the origin. Such a function satisfies the identity $f(-x) = -f(x)$ for all x. The name *odd* comes from the fact that $f(x) = x^n$ is an odd function whenever the exponent n is an odd integer.

operator notation: A method of naming a derivative by means of a prefix, usually D, as in $D \cos x = -\sin x$, or $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$, or $D_x(u^x) = u^x(\ln u)D_x u$.

orthonormal: Describes a set of mutually perpendicular vectors of unit length.

parabola: This curve consists of all the points that are equidistant from a given point (the *focus*) and a given line (the *directrix*).

parabolic method: A method of numerical integration that approximates the integrand by a piecewise-quadratic function.

paraboloid: One of the *quadric surfaces*. Sections obtained by slicing this surface with a plane that contains the principal axis are parabolas.

partial derivative: A *directional derivative* that is obtained by allowing only one of the variables to change.

partial sum: Given an infinite series $x_0+x_1+x_2+\cdots$, the finite series $x_0+x_1+x_2+\cdots+x_n$ is called the n^{th} *partial sum*.

path: A parametrization for a curve.

perihelion: The point on an orbit that is closest to the attracting focus.

polar coordinates: Polar coordinates for a point P in the xy-plane consist of two numbers r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray.

polar equation: An equation written using the polar variables r and θ.

potential function: An antiderivative for a *vector field*.

power series: A series of the form $\sum c_n(x-a)^n$. See also *Taylor series*.

Prime Number Theorem: See *logarithmic integral*.

prismoidal formula: To find the average value of a quadratic function on an interval, add two thirds of the value at the center to one sixth of the sum of the values at the endpoints.

Product Rule: The derivative of $p(x) = f(x)g(x)$ is $p'(x) = f(x)g'(x) + g(x)f'(x)$. The actual appearance of this rule depends on what x, f, g, and "product" mean, however. One can multiply numbers times numbers, numbers times vectors, and vectors times vectors — in two different ways.

quadric surface: The graph of a quadratic polynomial in three variables.

Quotient Rule: The derivative of $p(x) = \dfrac{f(x)}{g(x)}$ is $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$. This is unchanged in multivariable calculus, because vectors cannot be used as divisors.

radius of convergence: A *power series* $\Sigma c_n(x-a)^n$ converges for all x-values in an interval $a-r < x < a+r$ centered at a. The largest such r is the radius of convergence. It can be 0 or ∞.

radius of curvature: Given a point $\mathbf{p}$ on a differentiable curve, this is the distance from $\mathbf{p}$ to the *center of curvature* for that point.

Ratio Test: Provides a sufficient condition for the convergence of a positive series.

relative maximum means the same thing as *local maximum*.

relative minimum means the same thing as *local minimum*.

Rolle's Theorem: If f is a differentiable function, and $f(a) = 0 = f(b)$, then $f'(c) = 0$ for at least one c between a and b. Michel Rolle (1652-1719) described the emerging calculus as a collection of ingenious fallacies.

saddle point: Given a real-valued differentiable function f, this is a critical point $\mathbf{p}$ for f at which $f(\mathbf{p})$ is not extreme.

second derivative: The derivative of a derivative. If f is a real-valued function of $\mathbf{p}$, then $f'(\mathbf{p})$ is a vector that is usually called the *gradient* of f, and $f''(\mathbf{p})$ is a square matrix that is often called the *Hessian* of f. The entries in these arrays are *partial derivatives*.

Second-Derivative Test: When it succeeds, this theorem classifies a critical point for a differentiable function as a local maximum, a local minimum, or a saddle point (which in the one-variable case is called an inflection point). The theorem is inconclusive if the determinant of the second-derivative matrix is 0.

separable: A differential equation that can be written in the form $f(y)\dfrac{dy}{dx} = g(x)$.

simple harmonic motion: A sinusoidal function of time that models the movement of some physical objects, such as weights suspended from springs.

sinh: See *hyperbolic functions*.

speed: The magnitude of *velocity*. For a parametric curve $(x, y) = (f(t), g(t))$, it is given by the formula $\sqrt{(x')^2 + (y')^2}$. Notice that that this is *not* the same as dy/dx.

spherical coordinates: Points in three-dimensional space can be described as (ρ, θ, ϕ), where ρ is the distance to the origin, θ is longitude, and ϕ is co-latitude.

stereographic projection: Establishes a one-to-one correspondence between the points of a plane and the points of a punctured sphere, or between the points of a line and the points of a punctured circle.

tacking: A technical term used by sailors.

tanh: See *hyperbolic functions*.

Mathematics 6 Reference

Taylor polynomial: Given a differentiable function f, a Taylor polynomial $\sum c_n(x-a)^n$ matches all derivatives at $x=a$ through a given order. The coefficient of $(x-a)^n$ is given by Taylor's formula $c_n = \dfrac{1}{n!}f^{(n)}(a)$. Brook Taylor (1685-1731) wrote books on perspective, and re-invented Taylor series.

Taylor series: A *power series* $\sum c_n(x-a)^n$ in which the coefficients are calculated using Taylor's formula $c_n = \dfrac{1}{n!}f^{(n)}(a)$. The series is said to be "based at a."

Taylor's Theorem: The difference $f(b) - p_n(b)$ between a function f and its n^{th} Taylor polynomial is $\displaystyle\int_a^b f^{(n+1)}(x)\frac{1}{n!}(b-x)^n\,dx$.

telescope: Refers to infinite series whose partial sums happen to collapse.

torus: A surface that models an inner tube, or the boundary of a doughnut.

triangle inequality: The inequality $PQ \le PR + RQ$ says that any side of any triangle is at most equal to the sum of the other two sides.

triple scalar product: A formula for finding the volume of parallelepiped, in terms of its defining vectors. It is the *determinant* of a 3×3 matrix.

velocity: This n-dimensional vector is the derivative of a differentiable path in $\mathbf{R}^n$. When $n=2$, whereby a curve $(x,y) = (f(t), g(t))$ is described parametrically, the velocity is $\left[\dfrac{df}{dt}, \dfrac{dg}{dt}\right]$ or $\left[\dfrac{dx}{dt}, \dfrac{dy}{dt}\right]$, which is tangent to the curve. Its magnitude $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ is the speed. The *components* of velocity are themselves derivatives.

vector field: This is a descriptive name for a function F from $\mathbf{R}^n$ to $\mathbf{R}^n$. For each $\mathbf{p}$ in the domain, $F(\mathbf{p})$ is a vector. The derivative (gradient) of a real-valued function is an example of such a field.

Wallis product formula is $\dfrac{\pi}{2} = \dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdots = \displaystyle\lim_{k\to\infty}\left(\frac{4^k k!\,k!}{(2k)!}\right)^2\frac{1}{2k+1}$. It was published in 1655 by John Wallis (1616-1703), who made original contributions to calculus and geometry.

weighted average: A sum $p_1y_1 + p_2y_2 + p_3y_3 + \cdots + p_ny_n$ is called a *weighted average* of the numbers y_1, y_2, y_3,..., y_n, provided that $p_1 + p_2 + p_3 + \cdots + p_n = 1$ and each *weight* p_k is nonnegative. If $p_k = \dfrac{1}{n}$ for every k, this average is called the *arithmetic mean.*

zero: A number that produces 0 as a functional value. For example, $\sqrt{2}$ is one of the zeros of the function $f(x) = x^2 - 2$, and 1 is a zero of any logarithm function, because $\log 1$ is 0.

Mathematics 6 Appendix

Selected Derivative Formulas

1. $D_x au = a \cdot D_x u \qquad (a \text{ is constant})$

2. $D_x uv = u \cdot D_x v + v \cdot D_x u$

3. $D_x \dfrac{u}{v} = \dfrac{v \cdot D_x u - u \cdot D_x v}{v^2}$

4. $D_x u^n = nu^{n-1} \cdot D_x u$

5. $D_x \ln u = \dfrac{1}{u} \cdot D_x u$

6. $D_x \log_a u = (\log_a e)\, \dfrac{1}{u} \cdot D_x u$

7. $D_x e^u = e^u \cdot D_x u$

8. $D_x a^u = (\ln a) a^u \cdot D_x u$

9. $D_x \sin u = \cos u \cdot D_x u$

10. $D_x \cos u = -\sin u \cdot D_x u$

11. $D_x \tan u = \sec^2 u \cdot D_x u$

12. $D_x \cot u = -\csc^2 u \cdot D_x u$

13. $D_x \sec u = \sec u \tan u \cdot D_x u$

14. $D_x \csc u = -\csc u \cot u \cdot D_x u$

15. $D_x \sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}} \cdot D_x u$

16. $D_x \arcsin u = \dfrac{1}{\sqrt{1-u^2}} \cdot D_x u$

17. $D_x \cos^{-1} u = -\dfrac{1}{\sqrt{1-u^2}} \cdot D_x u$

18. $D_x \arccos u = -\dfrac{1}{\sqrt{1-u^2}} \cdot D_x u$

19. $D_x \tan^{-1} u = \dfrac{1}{1+u^2} \cdot D_x u$

20. $D_x \arctan u = \dfrac{1}{1+u^2} \cdot D_x u$

21. $D_x f(u) = D_u f(u) \cdot D_x u$

22. $D_x u^v = vu^{v-1} \cdot D_x u + u^v \ln u \cdot D_x v$

Mathematics 6 Appendix

Selected Antiderivative Formulas

1. $\displaystyle\int u^n\,du = \frac{1}{n+1}\,u^{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{u^n}\,du = \frac{1}{1-n}\,u^{1-n} + C \quad (n \neq 1)$

3. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

4. $\displaystyle\int a^u\,du = \frac{1}{\ln a}\,a^u + C$

5. $\displaystyle\int e^u\,du = e^u + C$

6. $\displaystyle\int \ln u\,du = u\ln u - u + C$

7. $\displaystyle\int \sin u\,du = -\cos u + C$

8. $\displaystyle\int \cos u\,du = \sin u + C$

9. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

10. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

11. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

12. $\displaystyle\int \csc u\,du = \ln\left|\tan\frac{u}{2}\right| + C$

13. $\displaystyle\int \sec u\,du = \frac{1}{2}\ln\left|\frac{1+\sin u}{1-\sin u}\right| + C$

14. $\displaystyle\int \csc u\,du = -\frac{1}{2}\ln\left|\frac{1+\cos u}{1-\cos u}\right| + C$

15. $\displaystyle\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\,\arctan\frac{u}{a} + C \quad (a \neq 0)$

16. $\displaystyle\int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\,\ln\left|\frac{u-a}{u+a}\right| + C \quad (a \neq 0)$

17. $\displaystyle\int \frac{mu+b}{u^2+a^2}\,du = \frac{m}{2}\ln\left|u^2+a^2\right| + \frac{b}{a}\arctan\frac{u}{a} + C \quad (a \neq 0)$

18. $\displaystyle\int \frac{1}{(u-a)(u-b)}\,du = \frac{1}{a-b}\,\ln\left|\frac{u-a}{u-b}\right| + C \quad (a \neq b)$

19. $\displaystyle\int \frac{mu+k}{(u-a)(u-b)}\,du = \frac{m}{a-b}\,\ln\frac{|u-a|^a}{|u-b|^b} + \frac{k}{a-b}\,\ln\left|\frac{u-a}{u-b}\right| + C \quad (a \neq b)$

20. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C \quad (a \neq 0)$

21. $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

22. $\displaystyle\int \frac{1}{\sqrt{u^2+a^2}}\,du = \ln\left|u+\sqrt{u^2+a^2}\right| + C$

23. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\,u\sqrt{a^2-u^2} + \frac{a^2}{2}\,\arcsin\frac{u}{a} + C \quad (a \neq 0)$

24. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\,u\sqrt{u^2-a^2} - \frac{a^2}{2}\,\ln\left|u+\sqrt{u^2-a^2}\right| + C \quad (a \neq 0)$

25. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{1}{2}\,u\sqrt{u^2+a^2} + \frac{a^2}{2}\,\ln\left|u+\sqrt{u^2+a^2}\right| + C$

26. $\displaystyle\int \frac{1}{u^3+a^3}\,du = \frac{1}{a^2}\left(\frac{1}{6}\ln\left|\frac{(u+a)^3}{u^3+a^3}\right| + \frac{1}{\sqrt{3}}\,\arctan\frac{2u-a}{a\sqrt{3}}\right) + C \quad (a \neq 0)$

27. $\displaystyle\int e^{au}\sin bu\,du = \frac{1}{a^2+b^2}\,e^{au}\,(a\sin bu - b\cos bu) + C$

28. $\displaystyle\int e^{au}\cos bu\,du = \frac{1}{a^2+b^2}\,e^{au}\,(a\cos bu + b\sin bu) + C$

29. $\displaystyle\int \sin au\sin bu\,du = \frac{1}{b^2-a^2}\,(a\cos au\sin bu - b\sin au\cos bu) + C \quad (a^2 \neq b^2)$

30. $\displaystyle\int \cos au\cos bu\,du = \frac{1}{b^2-a^2}\,(b\cos au\sin bu - a\sin au\cos bu) + C \quad (a^2 \neq b^2)$

31. $\displaystyle\int \sin au\cos bu\,du = \frac{1}{b^2-a^2}\,(b\sin au\sin bu + a\cos au\cos bu) + C \quad (a^2 \neq b^2)$

32. $\displaystyle\int u^n\ln u\,du = \frac{1}{n+1}\,u^{n+1}\ln|u| - \frac{1}{(n+1)^2}\,u^{n+1} + C \quad (n \neq -1)$

33. $\displaystyle\int \sin^n u\,du = -\frac{1}{n}\,\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du + C \quad (1 < n)$

34. $\displaystyle\int \cos^n u\,du = \frac{1}{n}\,\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du + C \quad (1 < n)$

Mathematics 6 Appendix

Hyperbolic Functions

$$\cosh t = \frac{e^t + e^{-t}}{2} \qquad \sinh t = \frac{e^t - e^{-t}}{2} \qquad \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

The Circular-Hyperbolic Analogy

$x^2 + y^2 = 1$	$x^2 - y^2 = 1$
$x = \cos t \qquad y = \sin t$	$x = \cosh t \qquad y = \sinh t$
$\cos^2 t + \sin^2 t = 1$	$\cosh^2 t - \sinh^2 t = 1$
$\cos(t+u) = \cos t \cos u - \sin t \sin u$	$\cosh(t+u) = \cosh t \cosh u + \sinh t \sinh u$
$\sin(t+u) = \sin t \cos u + \cos t \sin u$	$\sinh(t+u) = \sinh t \cosh u + \cosh t \sinh u$
$\sin 2t = 2 \sin t \cos t$	$\sinh 2t = 2 \sinh t \cosh t$
$\cos 2t = \cos^2 t - \sin^2 t$	$\cosh 2t = \cosh^2 t + \sinh^2 t$
$\cos^2 t = \frac{1}{2}\left(1 + \cos(2t)\right)$	$\cosh^2 t = -\frac{1}{2}\left(1 + \cosh(2t)\right)$
$\sin^2 t = \frac{1}{2}\left(1 - \cos(2t)\right)$	$\sinh^2 t = -\frac{1}{2}\left(1 - \cosh(2t)\right)$
$\frac{d}{dt} \cos t = -\sin t$	$\frac{d}{dt} \cosh t = \sinh t$
$\frac{d}{dt} \sin t = \cos t$	$\frac{d}{dt} \sinh t = \cosh t$
$\frac{d}{dt} \tan t = \frac{1}{\cos^2 t} = \sec^2 t$	$\frac{d}{dt} \tanh t = \frac{1}{\cosh^2 t} = \mathrm{sech}^2 t$
$\frac{d}{dt} \sec t = \sec t \tan t$	$\frac{d}{dt} \mathrm{sech} t = -\mathrm{sech} t \tanh t$
$\frac{d}{dt} \csc t = -\csc t \cot t$	$\frac{d}{dt} \mathrm{csch} t = -\mathrm{csch} t \coth t$
$\frac{d}{dt} \cot t = -\frac{1}{\sin^2 t} = -\csc^2 t$	$\frac{d}{dt} \mathrm{coth} t = -\frac{1}{\sinh^2 t} = -\mathrm{csch}^2 t$
$\frac{d}{dt} \cos^{-1} t = \frac{-1}{\sqrt{1-t^2}} \quad (-1 < t < 1)$	$\frac{d}{dt} \cosh^{-1} t = \frac{1}{\sqrt{t^2-1}} \quad (1 < t)$
$\frac{d}{dt} \sin^{-1} t = \frac{1}{\sqrt{1-t^2}} \quad (-1 < t < 1)$	$\frac{d}{dt} \sinh^{-1} t = \frac{1}{\sqrt{t^2+1}}$
$\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}$	$\frac{d}{dt} \tanh^{-1} t = \frac{1}{1-t^2} \quad (-1 < t < 1)$